# Trapezoidal and Simpson's rules

## Why numerical integration?

Some integrals cannot be evaluated using standard techniques. For example, $\int_1^2 e^{x^2}\,dx$ has no elementary antiderivative. In such cases, we use **numerical methods** to approximate the integral.

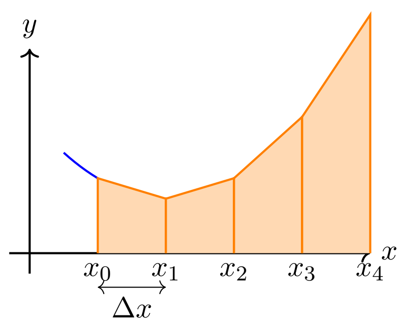
**Recall from Calculus I:** Left endpoint, right endpoint, and midpoint rules.

## Trapezoidal rule

The trapezoidal rule approximates the area under a curve using trapezoids instead of rectangles.

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.




**Pattern for coefficients:** $1, 2, 2, 2, \ldots, 2, 2, 1$

Example: Use the trapezoidal rule with $n = 4$ to approximate $\int_1^2 e^{x^2}\,dx$.

**Solution:** Here $a = 1$, $b = 2$, $n = 4$, so $\Delta x = \dfrac{2-1}{4} = 0.25$.

The $x$-values are: $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$.

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_i$ | 1 | 1.25 | 1.5 | 1.75 | 2 |
| $f(x_i) = e^{x_i^2}$ | 2.7183 | 4.7707 | 9.4877 | 21.1700 | 54.5982 |
| Coefficient | 1 | 2 | 2 | 2 | 1 |

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$$
\begin{aligned}
T_4 &= \frac{0.25}{2}\left[2.7183 + 2(4.7707) + 2(9.4877) + 2(21.1700) + 54.5982\right] \\
&= 0.125\left[2.7183 + 9.5414 + 18.9754 + 42.3400 + 54.5982\right] \\
&= 0.125(128.1733) \\
&\approx 16.022
\end{aligned}
$$

Note: Do NOT round intermediate results! Keep full precision until the final answer.

## Simpson's rule

Simpson's rule uses parabolic arcs instead of straight lines, giving a more accurate approximation.

$$
\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]
$$

where $n$ **must be even**, $\Delta x = \dfrac{b-a}{n}$, and $x_i = a + i\Delta x$.
**Pattern for coefficients:** $1, 4, 2, 4, 2, 4, \ldots, 4, 2, 4, 1$
Example: Use Simpson's rule with $n = 4$ to approximate $\int_1^2 e^{x^2}\,dx$.
**Solution:** Same $x$-values and function values as above. Only the coefficients change.

| $i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_i$ | 1 | 1.25 | 1.5 | 1.75 | 2 |
| $f(x_i) = e^{x_i^2}$ | 2.7183 | 4.7707 | 9.4877 | 21.1700 | 54.5982 |
| Coefficient | 1 | 4 | 2 | 4 | 1 |

$$
\begin{aligned}
S_4 &= \frac{0.25}{3}\left[2.7183 + 4(4.7707) + 2(9.4877) + 4(21.1700) + 54.5982\right] \\
&= \frac{0.25}{3}\left[2.7183 + 19.0828 + 18.9754 + 84.6800 + 54.5982\right] \\
&= \frac{0.25}{3}(180.0547) \\
&\approx 15.005
\end{aligned}
$$

## Working with data tables

When you have data (not a formula), apply the rules directly to the given values.
Example: The velocity of a car is recorded at half-hour intervals:

| Time $t$ (hr) | 2 | 2.5 | 3 | 3.5 |
|---|---|---|---|---|
| Velocity $v$ (km/hr) | 60 | 75 | 82 | 55 |

Estimate the total distance traveled using the trapezoidal rule.
**Solution:** Here $\Delta t = 0.5$ and $n = 3$.

$$\begin{aligned} \text{Distance} &\approx \frac{0.5}{2}\left[60 + 2(75) + 2(82) + 55\right] \\ &= 0.25\left[60 + 150 + 164 + 55\right] \\ &= 0.25(429) \\ &= 107.25 \text{ km} \end{aligned}$$

## Comparison of methods

| **Method** | **Coefficients** | **Divisor** | **Requirement** |
|---|---|---|---|
| Trapezoidal | $1, 2, 2, \ldots, 2, 1$ | $\frac{\Delta x}{2}$ | None |
| Simpson's | $1, 4, 2, 4, \ldots, 4, 1$ | $\frac{\Delta x}{3}$ | $n$ must be even |

**Accuracy comparison** for $\int_1^4 \frac{1}{x^2}\,dx = \frac{3}{4}$ (exact):

| **Method** | $n = 10$ | $n = 50$ | $n = 100$ |
|---|---|---|---|
| Right endpoint | $-0.0825$ | $-0.0175$ | $-0.0088$ |
| Left endpoint | $0.0953$ | $0.0180$ | $0.0090$ |
| Midpoint | $-0.0032$ | $-0.00013$ | $-0.00003$ |
| Trapezoidal | $0.0064$ | $0.00026$ | $0.00006$ |
| Simpson's | $0.00019$ | $3.4 \times 10^{-7}$ | $2.1 \times 10^{-8}$ |

**Error behaviour when $n$ increases by a factor of 10:**

- Left/Right endpoint: Error $\propto \frac{1}{n}$ (decreases by factor of 10)
- Midpoint/Trapezoidal: Error $\propto \frac{1}{n^2}$ (decreases by factor of 100)
- Simpson's: Error $\propto \frac{1}{n^4}$ (decreases by factor of 10,000)

## Error bounds

If $|f''(x)| \leq K$ for $a \leq x \leq b$, then:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

If $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$, then:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example: Find the maximum error when approximating $\displaystyle\int_0^{\pi/2} \cos x \, dx$ using Simpson's rule with $n = 4$.

**Solution:** We need $|f^{(4)}(x)| \leq K$ on $[0, \pi/2]$.

Since $f(x) = \cos x$, we have $f^{(4)}(x) = \cos x$. On $[0, \pi/2]$, the maximum value is $|\cos 0| = 1$, so $K = 1$.

$$|E_S| \leq \frac{1 \cdot (\pi/2)^5}{180 \cdot 4^4} = \frac{(\pi/2)^5}{180 \cdot 256} = \frac{\pi^5}{32 \cdot 180 \cdot 256} \approx 0.00066$$

Example: How large must $n$ be to guarantee the midpoint rule approximation of $\displaystyle\int_1^3 \frac{1}{x} \, dx$ has error less than $10^{-8}$?

**Solution:** For $f(x) = \dfrac{1}{x}$, we have $f''(x) = \dfrac{2}{x^3}$. On $[1, 3]$, this is maximized at $x = 1$, so $K = 2$.

We need:

$$\frac{2(3-1)^3}{24n^2} < 10^{-8} \implies \frac{16}{24n^2} < 10^{-8} \implies n^2 > \frac{16 \times 10^8}{24} \implies n > \sqrt{\frac{2 \times 10^8}{3}} \approx 8165$$

So we need $n \geq 8166$.

Note: Always round $n$ **up** to ensure the error bound is satisfied.

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