Statistical quality control

Overview

A **control chart** monitors a process over time by plotting a summary statistic from successive samples against the sample number. Two horizontal lines — the **upper control limit (UCL)** and **lower control limit (LCL)** — bound the region of expected variation. Points outside these limits signal a possible **special cause**.

Quality characteristic	Chart type	Charts used
Continuous variable	Variables control chart	$\bar{X}$ chart, R chart, S chart
Binary (defective/not)	Attribute control chart	p chart
Count of defects	Attribute control chart	c chart

Key notation: k = number of samples (subgroups); n = subgroup size (items per sample).
Note: Always construct the **variation chart** (R or S) before the mean chart ($\bar{X}$). The centerline of the variation chart is needed to compute the $\bar{X}$ chart limits.

The R chart (sample range)

The R chart monitors process variability using the range of each subgroup.

$$\bar{R} = \frac{1}{k}\sum_{i=1}^{k} R_i \qquad \text{UCL} = D_4\bar{R} \qquad \text{LCL} = D_3\bar{R}$$

Constants D_3 and D_4 depend on subgroup size n (from control chart constants table). The process standard deviation can be estimated as $\hat{\sigma} = \bar{R}/d_2$.

The S chart (sample standard deviation)

The S chart is an alternative to the R chart, also used to monitor process variability. It uses the standard deviation of each subgroup rather than the range. For large n, it is more precise than the R chart.

$$\bar{s} = \frac{1}{k}\sum_{i=1}^{k} s_i \qquad \text{UCL} = B_4\bar{s} \qquad \text{LCL} = B_3\bar{s}$$

Constants B_3 and B_4 depend on n (from table). The process standard deviation can be estimated as $\hat{\sigma} = \bar{s}/c_4$.

The $\bar{X}$ chart (sample mean)

Once the R or S chart is in control, the $\bar{X}$ chart monitors the process mean.
Using R:

$$\bar{\bar{x}} = \frac{1}{k}\sum_{i=1}^{k} \bar{x}_i \qquad \text{UCL} = \bar{\bar{x}} + A_2\bar{R} \qquad \text{LCL} = \bar{\bar{x}} - A_2\bar{R}$$

Using S:

$$\text{UCL} = \bar{\bar{x}} + A_3\bar{s} \qquad \text{LCL} = \bar{\bar{x}} - A_3\bar{s}$$

Constants A_2 and A_3 depend on n (from table).

Workflow for variables control charts

1. Choose rational subgroups (variation within each subgroup should reflect only common causes).
2. Compute the R or S chart. Identify and investigate any out-of-control points.
3. Remove samples associated with out-of-control points and recompute the R or S chart.
4. Once the variation chart is in control, compute the $\bar{X}$ chart (omitting the same removed samples).
5. If the $\bar{X}$ chart shows out-of-control points, identify and correct the special causes.
6. Continue monitoring both charts.

Average run length (ARL)

A **false alarm** occurs when a point plots outside the control limits even though the process is in control. A **failure to detect** occurs when the process is out of control but no point plots outside the limits.
The **average run length** is the expected number of samples observed before a point plots outside the control limits:

$$\text{ARL} = \frac{1}{p}$$

where p is the probability that any given point plots outside the control limits.
Example: A process has $\mu = 3$, $\sigma = 1$, and samples of size $n = 4$. A special cause shifts the mean to $\mu = 3.5$. Find the ARL.
Solution: The $\bar{X}$ chart has 3-sigma limits based on $\sigma_{\bar{x}} = \sigma/\sqrt{n} = 1/2 = 0.5$:

$$\text{UCL} = 3 + 3(0.5) = 4.5 \qquad \text{LCL} = 3 - 3(0.5) = 1.5$$

After the shift, $\bar{X} \sim N(3.5,\, 0.5^2)$. The probability of a point falling outside the limits:

$$p = P(\bar{X} > 4.5) + P(\bar{X} < 1.5) = P\left(Z > \frac{4.5 - 3.5}{0.5}\right) + P\left(Z < \frac{1.5 - 3.5}{0.5}\right) = P(Z > 2) + P(Z < -4)$$

$$p \approx 0.0228 + 0 = 0.0228 \qquad \text{ARL} = \frac{1}{0.0228} \approx 43.9 \text{ samples}$$

The p chart (proportion defective)

Used when each unit is classified as **defective or not** (binary). Let $\hat{p}_i = x_i/n$ be the proportion defective in sample i.

$$\bar{p} = \frac{1}{k}\sum_{i=1}^{k} \hat{p}_i \qquad \text{UCL} = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \qquad \text{LCL} = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

Note: If the computed LCL is negative, set LCL $= 0$.

Interpreting out-of-control points on a p chart:

- Point **above the UCL**: the proportion of defectives has increased. Identify and **remove** the special cause.
- Point **below the LCL**: the proportion of defectives has decreased. Identify the special cause and **make it permanent** to sustain the improvement.

The c chart (count of defects)

Used when the quality measurement is a **count of defects** in a unit. Requires that defects follow a **Poisson distribution**. Let c_i be the number of defects in unit i and $\bar{c}$ be the average.

$$\bar{c} = \frac{1}{k}\sum_{i=1}^{k} c_i \qquad \text{UCL} = \bar{c} + 3\sqrt{\bar{c}} \qquad \text{LCL} = \bar{c} - 3\sqrt{\bar{c}}$$

Note: If the computed LCL is negative, set LCL $= 0$. The same interpretation of above/below the control limits applies as with the p chart.

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Summary of control chart formulas

Chart	Centerline	UCL	LCL
R	$\bar{R}$	$D_4\bar{R}$	$D_3\bar{R}$
S	$\bar{s}$	$B_4\bar{s}$	$B_3\bar{s}$
$\bar{X}$ (using R)	$\bar{\bar{x}}$	$\bar{\bar{x}} + A_2\bar{R}$	$\bar{\bar{x}} - A_2\bar{R}$
$\bar{X}$ (using S)	$\bar{\bar{x}}$	$\bar{\bar{x}} + A_3\bar{s}$	$\bar{\bar{x}} - A_3\bar{s}$
p	$\bar{p}$	$\bar{p} + 3\sqrt{\bar{p}(1-\bar{p})/n}$	$\bar{p} - 3\sqrt{\bar{p}(1-\bar{p})/n}$
c	$\bar{c}$	$\bar{c} + 3\sqrt{\bar{c}}$	$\bar{c} - 3\sqrt{\bar{c}}$

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