Stars and bars

The idea

Stars and bars counts the number of ways to distribute n **identical** objects into k **distinct** bins. It also counts the non-negative integer solutions to equations of the form $x_1 + x_2 + \cdots + x_k = n$. These are the same problem.

The trick: represent a distribution as a row of n stars ($\star$) and $k-1$ bars ($|$). The bars divide the stars into k groups.

Arrangement: $\star \ \star \ \star \ | \ \star \ \star \ | \quad | \ \star \ \star \ \star$

Bin totals: Bin 1: 3   Bin 2: 2   Bin 3: 0   Bin 4: 3

Every arrangement of n stars and $k-1$ bars corresponds to exactly one distribution, and vice versa. So the problem reduces to: **in how many ways can we choose the positions of the $k-1$ bars** among the $n+k-1$ total symbols?

The two formulas

Condition	Formula	Use when...
Bins may be empty ($x_i \geq 0$)	$\binom{n+k-1}{k-1}$	No minimum requirement per bin
Each bin gets ≥ 1 ($x_i \geq 1$)	$\binom{n-1}{k-1}$	Every bin must be non-empty

Note: $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$—both forms appear in textbooks and are equal.

Examples

Example 1: How many ways can 8 identical candies be distributed among 4 children (some children may receive none)?

Solution: $n = 8$ candies, $k = 4$ children, no restriction (bins may be empty):

$$\binom{8+4-1}{4-1} = \binom{11}{3} = \frac{11 \cdot 10 \cdot 9}{3!} = 165$$

Example 2: How many non-negative integer solutions does $x_1 + x_2 + x_3 = 10$ have?
Solution: Distributing 10 identical units among 3 variables ($k = 3$, $n = 10$):

$$\binom{10+3-1}{3-1} = \binom{12}{2} = 66$$

Example 3: How many ways can 8 identical candies be distributed among 4 children so that **each child gets at least one**?
Solution: Give 1 candy to each child first (using up $4 \times 1 = 4$ candies). Distribute the remaining $8 - 4 = 4$ candies freely among the 4 children:

$$\binom{4+4-1}{4-1} = \binom{7}{3} = 35$$

Equivalently, use the direct formula with $n - k = 8 - 4 = 4$ substituted in: $\binom{n-1}{k-1} = \binom{7}{3} = 35$.

Handling a minimum greater than 1

If each bin must contain at least m objects, substitute $x_i' = x_i - m$ (so $x_i' \geq 0$). The new total becomes $n - mk$, and you apply the no-restriction formula.
Example 4: How many integer solutions does $x_1 + x_2 + x_3 = 14$ have with $x_i \geq 2$ for all i?
Solution: Let $x_i' = x_i - 2$, so $x_i' \geq 0$. Substituting:

$$(x_1' + 2) + (x_2' + 2) + (x_3' + 2) = 14 \implies x_1' + x_2' + x_3' = 8$$

$$\binom{8+3-1}{3-1} = \binom{10}{2} = 45$$

Handling an upper bound

If a bin can hold **at most** u objects, use inclusion–exclusion: count all distributions, then subtract those where any bin exceeds u.
Example 5: How many ways can 7 identical balls be put into 3 boxes with each box holding **at most 4**?
Solution: Total (no restriction): $\binom{9}{2} = 36$.
Subtract cases where some box gets ≥ 5: let box i take $5+$ balls. Set $x_i' = x_i - 5 \geq 0$; remaining total is $7 - 5 = 2$, distributed in $\binom{4}{2} = 6$ ways. There are 3 choices for which box overflows, giving $3 \times 6 = 18$.
Check for double overcounting: two boxes each getting ≥ 5 would need ≥ 10 balls total, impossible here. So no correction needed.

$$36 - 18 = \mathbf{18}$$

Common mistakes

Caution:

- **Objects must be identical.** Stars and bars does not apply when objects are distinguishable—use permutations or the multiplication rule instead.
- **Count the bars, not the bins.** You need $k-1$ bars for k bins; choosing their positions from $n+k-1$ slots gives $\binom{n+k-1}{k-1}$.
- **Apply the substitution before using the formula.** For $x_i \geq m$, always reduce n by mk first.

Practice problems.

1. How many ways can 12 identical books be placed on 5 distinct shelves (shelves may be empty)?

2. How many non-negative integer solutions does $x_1 + x_2 + x_3 + x_4 = 9$ have?

3. How many ways can 12 identical books be placed on 5 shelves so that **each shelf has at least one book**?

4. How many integer solutions does $x_1 + x_2 + x_3 = 18$ have with $x_i \geq 3$ for all i?

5. A convenience store sells 6 types of chocolate bar. How many ways can you choose 10 bars (repetition allowed, order doesn't matter)?

6. How many ways can 10 identical balls be placed into 4 boxes with each box holding **at most 4**?

Answers: 1. $\binom{16}{4} = 1820$ 2. $\binom{12}{3} = 220$ 3. $\binom{11}{4} = 330$ 4. $x_i' = x_i - 3$: $\binom{11}{2} = 55$ 5. $\binom{15}{5} = 3003$ 6. Total $\binom{13}{3} = 286$; subtract $4\binom{8}{3} = 224$; add back $\binom{4}{2}\binom{3}{3} = 6$: answer $= 68$

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