Solving for a variable

The goal

Solving for a variable means isolating it on one side of the equation. Whatever operation is applied to one side **must also be applied to the other side** to keep the equation balanced.

Reverse BEDMAS

When an expression has been built up by following order of operations (BEDMAS), we undo it by **working backwards through BEDMAS**. The last operation applied to the variable is the first one we undo:

Step	Undo...	By...
1	Addition or subtraction	Subtracting or adding
2	Multiplication or division	Dividing or multiplying
3	Exponents (x^n)	Taking the nth root
4	Roots ($\sqrt[n]{x}$)	Raising to the power n
5	Brackets	Expanding first, then Steps 1–4
6	Exponential functions (e^x, a^x)	Taking a logarithm
7	Logarithms ($\ln x$, $\log_b x$)	Exponentiating
8	Trig functions (sin, cos, tan)	Applying the inverse trig function

Note: If the equation has brackets, **expand them first** before applying the reverse BEDMAS steps. After expanding, start at Step 1.

Step 1: Undoing addition and subtraction

Addition and subtraction are undone by applying the opposite operation to both sides.
Example: Solve $x + 7 = 15$.

$$x = 15 - 7 = 8$$

Example: Solve $x - 4 = -1$.

$$x = -1 + 4 = 3$$

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Step 2: Undoing multiplication and division

Multiplication is undone by dividing; division is undone by multiplying.
Example: Solve $5x = 30$.

$$x = \frac{30}{5} = 6$$

Example: Solve $\frac{x}{4} = 3$.

$$x = 3 \times 4 = 12$$

Example: Solve $3x - 5 = 16$. *(Steps 1 then 2)*

$$3x = 21 \quad \text{(add 5 — undo subtraction first)}$$
$$x = 7 \quad \text{(divide by 3 — undo multiplication second)}$$

Step 3: Undoing exponents

An exponent x^n is undone by taking the nth root of both sides.
Example: Solve $x^2 = 49$.

$$x = \pm\sqrt{49} = \pm 7$$

Note: When undoing an **even** exponent, include $\pm$: both 7 and -7 satisfy $x^2 = 49$. Odd exponents yield only one real solution.
Example: Solve $2x^3 = 54$.

$$x^3 = 27 \quad \text{(divide by 2)}$$
$$x = \sqrt[3]{27} = 3$$

Step 4: Undoing roots

A root $\sqrt[n]{x}$ is undone by raising both sides to the power n.
Example: Solve $\sqrt{x} = 6$.

$$x = 6^2 = 36$$

Example: Solve $\sqrt{x - 3} = 4$.

$$x - 3 = 16 \quad \text{(square both sides — undo the root)}$$
$$x = 19 \quad \text{(add 3 — undo the subtraction)}$$

Caution: After squaring both sides, always check your answer in the original equation. Squaring can introduce **extraneous solutions** that do not actually satisfy the original.

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Step 5: Undoing brackets

Expand brackets using the distributive law first, then apply Steps 1–4.

Example: Solve $4(x-2) = 3x + 1$.

$$4x - 8 = 3x + 1 \quad \text{(expand)}$$
$$x = 9 \quad \text{(subtract } 3x\text{, then add 8)}$$

Example: Solve $3(2x+1) - 4 = 2(x+5)$.

$$6x + 3 - 4 = 2x + 10 \quad \text{(expand both)}$$
$$6x - 1 = 2x + 10$$
$$4x = 11$$
$$x = \frac{11}{4}$$

Step 6: Undoing exponential functions

An exponential a^x or e^x is undone by taking a logarithm of both sides. Use ln (natural log) to undo e^x, and $\log_a$ to undo a^x. Then apply the power rule $\ln(a^x) = x \ln a$ to bring the exponent down.

Exponential form	How to undo
$e^x = k$	Take ln of both sides: $x = \ln k$
$a^x = k$	Take ln of both sides: $x \ln a = \ln k$, so $x = \frac{\ln k}{\ln a}$

Example: Solve $e^x = 12$.

$$\ln(e^x) = \ln 12 \implies x = \ln 12 \approx 2.485$$

Example: Solve $3e^x - 1 = 20$.

$$3e^x = 21 \quad \text{(add 1)}$$
$$e^x = 7 \quad \text{(divide by 3)}$$
$$x = \ln 7 \approx 1.946$$

Example: Solve $5^x = 80$.

$$\ln(5^x) = \ln 80$$
$$x \ln 5 = \ln 80 \quad \text{(power rule: bring down the exponent)}$$
$$x = \frac{\ln 80}{\ln 5} \approx \frac{4.382}{1.609} \approx 2.723$$

Example: Solve $2 \cdot 4^{3x} = 128$.

$$4^{3x} = 64 \quad \text{(divide by 2)}$$
$$3x \ln 4 = \ln 64 \quad \text{(take ln; power rule)}$$
$$x = \frac{\ln 64}{3 \ln 4} = \frac{3 \ln 4}{3 \ln 4} = 1$$

Note: In the last example, recognizing that $64 = 4^3$ allows the logarithm step to be avoided entirely. Always check whether the base and result are related by a clean power before reaching for ln.

Step 7: Undoing logarithms

A logarithm $\log_b x$ is undone by exponentiating both sides with base b. This uses the inverse relationship $b^{\log_b x} = x$.

Logarithmic form	**How to undo**
$\ln x = k$	Exponentiate: $x = e^k$
$\log x = k$	Exponentiate base 10: $x = 10^k$
$\log_b x = k$	Exponentiate base b: $x = b^k$

Example: Solve $\ln x = 4$.

$$x = e^4 \approx 54.60$$

Example: Solve $\ln(x - 2) = 3$.

$$x - 2 = e^3 \quad \text{(exponentiate both sides)}$$
$$x = e^3 + 2 \approx 22.09$$

Example: Solve $2 \log_3 x + 1 = 7$.

$$2 \log_3 x = 6 \quad \text{(subtract 1)}$$
$$\log_3 x = 3 \quad \text{(divide by 2)}$$
$$x = 3^3 = 27$$

Note: After solving, verify that the argument of the logarithm is positive in the original equation. Logarithms are undefined for non-positive inputs, so extraneous solutions are possible.

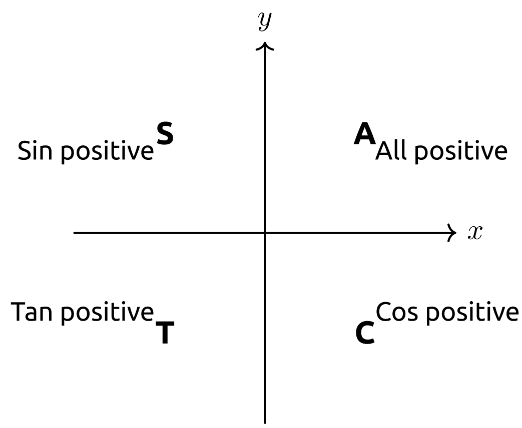
Step 8: Undoing trigonometric functions

Because trig functions are **periodic**, an equation like $\sin x = k$ has infinitely many solutions. We find them in three steps:

1. **Find the principal angle.** Apply the inverse trig function to get the reference solution. This gives one angle in the restricted range of that inverse function.
2. **Add enough terms.** Trig functions repeat, so add the appropriate multiples of the period to the principal angle to reach all solutions (or all solutions in a given interval).
3. **Apply the CAST rule.** Use the CAST rule to identify which other angle in the current period shares the same trig value, and include it.

The CAST rule

The CAST rule describes which trig functions are **positive** in each quadrant:



Given the principal angle $\alpha = \text{arcsin}(k)$, $\text{arccos}(k)$, or $\text{arctan}(k)$, the CAST rule tells you which other quadrant contains a second solution in the same period:

Function	Principal angle	Second solution in $[0, 2\pi)$
$\sin x = k$	$\alpha = \text{arcsin}(k)$	$\pi - \alpha$ (S and A quadrants both have positive sin)
$\cos x = k$	$\alpha = \text{arccos}(k)$	$2\pi - \alpha$ (C and A quadrants both have positive cos)
$\tan x = k$	$\alpha = \text{arctan}(k)$	$\pi + \alpha$ (T and A quadrants both have positive tan)

Note: If k is negative, the principal angle will land outside the first quadrant. Use the CAST rule to confirm which quadrants apply, and adjust accordingly.

Example: Solve $2\sin x - 1 = 0$ for $x \in [0, 2\pi)$.

Step 1 — Isolate and find the principal angle:

$$\sin x = \frac{1}{2}$$
$$x = \text{arcsin}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

Step 2 — Add enough terms: We need solutions in $[0, 2\pi)$, so one period is enough.

Step 3 — CAST rule: Since $\sin x > 0$, solutions lie in quadrants A and S. The second solution is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

$$x = \frac{\pi}{6}, \quad x = \frac{5\pi}{6}$$

Example: Solve $\cos x = -\frac{\sqrt{2}}{2}$ for $x \in [0, 2\pi)$.

Step 1 — Principal angle: $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ (using the positive value as the reference).

Step 2 — Add enough terms: One period suffices for $[0, 2\pi)$.

Step 3 — CAST rule: Since $\cos x < 0$, solutions lie in quadrants S and T. The two solutions are:

$$x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}, \qquad x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

Practice problems. Solve for x. Give exact answers where possible; otherwise round to three decimal places.

Exponents, roots, and brackets.

1. $4x - 3 = 13$ **2.** $3(x+2) = 2x + 11$ **3.** $x^2 = 81$ **4.** $3x^2 - 12 = 63$

5. $x^3 = -27$ **6.** $\sqrt{x+4} = 5$ **7.** $2\sqrt{x} - 3 = 7$ **8.** $5(x-1) = 3(x+3)$

Exponentials and logarithms.

9. $e^x = 30$ **10.** $2e^x + 5 = 19$ **11.** $3^x = 45$ **12.** $4 \cdot 2^x = 64$

13. $\ln x = 5$ **14.** $\ln(x+1) = 2$ **15.** $\log_2 x = 4$ **16.** $3 \log x - 1 = 5$

Trigonometric equations. Solve for $x \in [0, 2\pi)$ unless otherwise stated.

17. $\sin x = \frac{\sqrt{3}}{2}$ **18.** $\cos x = \frac{1}{2}$ **19.** $\tan x = 1$ **20.** $2\cos x - \sqrt{3} = 0$

21. $\sin x = -\frac{1}{2}$ **22.** $\cos x = -1$ **23.** $\tan x = -\sqrt{3}$ **24.** $2\sin x + \sqrt{3} = 0$

Answers:

1. $x = 4$ 2. $x = 5$ 3. $x = \pm 9$ 4. $x = \pm 5$ 5. $x = -3$ 6. $x = 21$ 7. $x = 25$ 8. $x = 7$
9. $x = \ln 30 \approx 3.401$ 10. $x = \ln 7 \approx 1.946$ 11. $x = \frac{\ln 45}{\ln 3} \approx 3.465$ 12. $x = 4$ 13. $x = e^5 \approx 148.413$ 14. $x = e^2 - 1 \approx 6.389$ 15. $x = 16$ 16. $x = 100$
17. $x = \frac{\pi}{3}, \frac{2\pi}{3}$ 18. $x = \frac{\pi}{3}, \frac{5\pi}{3}$ 19. $x = \frac{\pi}{4}, \frac{5\pi}{4}$ 20. $x = \frac{\pi}{6}, \frac{11\pi}{6}$ 21. $x = \frac{7\pi}{6}, \frac{11\pi}{6}$ 22. $x = \pi$ 23. $x = \frac{2\pi}{3}, \frac{5\pi}{3}$ 24. $x = \frac{4\pi}{3}, \frac{5\pi}{3}$

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