Sequences and series

Definitions

A **sequence** is an ordered list of terms: $\{a_1, a_2, a_3, \ldots\}$ or $\{a_n\}_{n=1}^{\infty}$

A **series** is the sum of terms of a sequence: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

Convergence of sequences

A sequence $\{a_n\}$ **converges** to L if $\lim_{n\to\infty} a_n = L$.
If the limit does not exist (including $\pm\infty$), the sequence **diverges**.

Convergence tests for series

Test	When to use	Conclusion
Divergence Test	Any series	If $\lim_{n\to\infty} a_n \neq 0$, series diverges If $\lim_{n\to\infty} a_n = 0$, inconclusive
Geometric Series	$\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$	Converges to $\frac{a}{1-r}$ if $\|r\| < 1$ Diverges if $\|r\| \geq 1$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$ Diverges if $p \leq 1$
Ratio Test	Series with factorials and/or exponentials	Let $L = \lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$ $L < 1$: converges absolutely $L > 1$: diverges $L = 1$: inconclusive
Root Test	Series with n-th powers	Let $L = \lim_{n\to\infty} \sqrt[n]{\|a_n\|}$ $L < 1$: converges absolutely $L > 1$: diverges $L = 1$: inconclusive

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Examples

Example 1: $\sum_{k=0}^{\infty} \frac{2^k}{e^k}$

$$\sum_{k=0}^{\infty} \frac{2^k}{e^k} = \sum_{k=0}^{\infty} \left(\frac{2}{e}\right)^k.$$

This is a geometric series with $a = 1$ and $r = \frac{2}{e}$.
Since $\left|\frac{2}{e}\right| < 1$, the series converges and

$$\sum_{k=0}^{\infty} \left(\frac{2}{e}\right)^k = \frac{1}{1-\frac{2}{e}} = \frac{e}{e-2}.$$

Example 2: $\sum_{n=1}^{\infty} n^2 e^{-n}$

Use the **Ratio Test** (exponential present):

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{1}{e} = 1 \cdot \frac{1}{e} = \frac{1}{e} < 1$$

The series **converges absolutely**.

Example 3: $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

This is a p-**series** with $p = \frac{2}{3}$.
Since $p = \frac{2}{3} \leq 1$, the series **diverges**.

Example 4: $\sum_{n=0}^{\infty} \left(\frac{3n-3}{9n+4}\right)^{n+5}$

Use the **Root Test** (n-th power structure):

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n-3}{9n+4}\right)^{n+5}} = \lim_{n\to\infty} \left(\frac{3n-3}{9n+4}\right)^{(n+5)/n}$$

$$= \lim_{n\to\infty} \left(\frac{3n-3}{9n+4}\right)^1 = \lim_{n\to\infty} \frac{3n-3}{9n+4} = \frac{3}{9} = \frac{1}{3} < 1$$

The series **converges absolutely**.

Example 5: Determine whether the sequence $a_n = \frac{5n^2+1}{2n^2-7}$ converges or diverges.

Compute the limit:

$$\lim_{n\to\infty} \frac{5n^2+1}{2n^2-7} = \lim_{n\to\infty} \frac{5+\frac{1}{n^2}}{2-\frac{7}{n^2}} = \frac{5}{2}.$$

Therefore, the sequence **converges** to $\boxed{\frac{5}{2}}$.

Quick reference: Choosing a test

If you see...	**Try...**
ar^n or ratio of exponentials	Geometric Series
$\frac{1}{n^p}$	p-Series
Factorials ($n!$)	Ratio Test
Exponentials (a^n, e^n)	Ratio Test
n-th powers like $(f(n))^n$	Root Test
None of the above	Divergence Test first

Tip: Always check the Divergence Test first. If $\lim_{n\to\infty} a_n \neq 0$, the series diverges. But if the limit is 0, you must use another test.

Practice problems. Determine whether each sequence or series converges or diverges. State the test used for series.

1. $a_n = \dfrac{4n^2 - 1}{n^2 + 3}$

2. $a_n = \dfrac{(-1)^n}{n}$

3. $a_n = \dfrac{e^n}{n+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{5^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^2}$

8. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{5n}\right)^n$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Answers: 1. Converges to 4 2. Converges to 0 3. Sequence diverges 4. Converges (p-series) 5. Converges to 9 (geometric) 6. Diverges (Ratio Test) 7. Diverges (Divergence Test) 8. Converges (Root Test) 9. Diverges (Divergence Test) 10. Converges (Ratio Test)