

# Rules of inference



## Arguments

An **argument** is a sequence of propositions (premises) leading to a final proposition (conclusion).

A **valid argument** is one where the truth of the premises guarantees the truth of the conclusion.

$$\begin{array}{l}
 p_1 \\
 p_2 \\
 \vdots \\
 p_n \\
 \hline
 \therefore q
 \end{array}
 \quad \text{means} \quad (p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q \text{ is a tautology}$$

## Rules of inference for propositional logic

Name	Rule	Tautology
Modus Ponens	$  \begin{array}{l}  p \\  p \rightarrow q \\  \hline  \therefore q  \end{array}  $	$[p \wedge (p \rightarrow q)] \rightarrow q$
Modus Tollens	$  \begin{array}{l}  \neg q \\  p \rightarrow q \\  \hline  \therefore \neg p  \end{array}  $	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$
Hypothetical Syllogism	$  \begin{array}{l}  p \rightarrow q \\  q \rightarrow r \\  \hline  \therefore p \rightarrow r  \end{array}  $	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
Disjunctive Syllogism	$  \begin{array}{l}  p \vee q \\  \neg p \\  \hline  \therefore q  \end{array}  $	$[(p \vee q) \wedge \neg p] \rightarrow q$
Addition	$  \begin{array}{l}  p \\  \hline  \therefore p \vee q  \end{array}  $	$p \rightarrow (p \vee q)$
Simplification	$  \begin{array}{l}  p \wedge q \\  \hline  \therefore p  \end{array}  $	$(p \wedge q) \rightarrow p$
Conjunction	$  \begin{array}{l}  p \\  q \\  \hline  \therefore p \wedge q  \end{array}  $	$[(p) \wedge (q)] \rightarrow (p \wedge q)$
Resolution	$  \begin{array}{l}  p \vee q \\  \neg p \vee r \\  \hline  \therefore q \vee r  \end{array}  $	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

## How to remember them

**Modus Ponens:** "If  $p$  then  $q$ ;  $p$  is true; therefore  $q$ ." (Affirm the antecedent)

**Modus Tollens:** "If  $p$  then  $q$ ;  $q$  is false; therefore  $p$  is false." (Deny the consequent)

**Hypothetical Syllogism:** Chain implications together.

**Disjunctive Syllogism:** " $p$  or  $q$ ; not  $p$ ; therefore  $q$ ." (Process of elimination)

## Example: Identifying rules

**Example:** Which rule of inference is used?

"Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard."

Let  $p$  = "Linda is an excellent swimmer" and  $q$  = "Linda can work as a lifeguard."

The argument is:  $p, p \rightarrow q$ , therefore  $q$ .

This is **Modus Ponens**.

## Example: Building an argument

**Example:** Show that  $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s, s \rightarrow t$  imply  $t$ .

Step	Statement	Reason
1	$\neg p \wedge q$	Premise
2	$\neg p$	Simplification from (1)
3	$r \rightarrow p$	Premise
4	$\neg r$	Modus Tollens from (2), (3)
5	$\neg r \rightarrow s$	Premise
6	$s$	Modus Ponens from (4), (5)
7	$s \rightarrow t$	Premise
8	$t$	Modus Ponens from (6), (7)

Therefore, the conclusion  $t$  follows from the premises. □

## Rules of inference for quantified statements

Name	Rule	Meaning
Universal Instantiation	$\frac{\forall x P(x)}{\therefore P(c)}$	If true for all, true for any specific $c$
Universal Generalization	$\frac{P(c) \text{ for arbitrary } c}{\therefore \forall x P(x)}$	If true for arbitrary $c$ , true for all
Existential Instantiation	$\frac{\exists x P(x)}{\therefore P(c) \text{ for some } c}$	If one exists, give it a name $c$
Existential Generalization	$\frac{P(c) \text{ for some } c}{\therefore \exists x P(x)}$	If true for some $c$ , then one exists

**Caution:** For Universal Generalization,  $c$  must be **arbitrary**—you cannot have assumed anything special about  $c$ .

### Example: Quantified argument

**Example:** Show the argument is valid:

“Everyone in New Jersey lives within 50 miles of the ocean.”

“Someone in New Jersey has not seen the ocean.”

“Therefore, someone who lives within 50 miles of the ocean has not seen it.”

Let  $N(x)$  = “ $x$  is in New Jersey,”  $O(x)$  = “ $x$  lives within 50 miles of ocean,”  $S(x)$  = “ $x$  has seen the ocean.”

Step	Statement	Reason
1	$\forall x [N(x) \rightarrow O(x)]$	Premise 1
2	$\exists x [N(x) \wedge \neg S(x)]$	Premise 2
3	$N(c) \wedge \neg S(c)$	Existential Instantiation from (2)
4	$N(c)$	Simplification from (3)
5	$N(c) \rightarrow O(c)$	Universal Instantiation from (1)
6	$O(c)$	Modus Ponens from (4), (5)
7	$\neg S(c)$	Simplification from (3)
8	$O(c) \wedge \neg S(c)$	Conjunction from (6), (7)
9	$\exists x [O(x) \wedge \neg S(x)]$	Existential Generalization from (8)

The conclusion follows. □

Student Learning Support, Teaching and Learning Centre

[studentlearning@ontariotechu.ca](mailto:studentlearning@ontariotechu.ca)

[ontariotechu.ca/studentlearning](http://ontariotechu.ca/studentlearning)



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## Common fallacies

Fallacy	Invalid form	Why it fails
Affirming the Consequent	$\frac{q}{p \rightarrow q}$ $\therefore p$	$q$ could be true for other reasons
Denying the Antecedent	$\frac{\neg p}{p \rightarrow q}$ $\therefore \neg q$	$q$ could still be true

**Example of fallacy:** “If it rains, the ground is wet. The ground is wet. Therefore, it rained.”  
This is **Affirming the Consequent**—invalid because the sprinklers could have run!

## Quick reference

If you have...	Use...
$p$ and $p \rightarrow q$	Modus Ponens to get $q$
$\neg q$ and $p \rightarrow q$	Modus Tollens to get $\neg p$
$p \rightarrow q$ and $q \rightarrow r$	Hypothetical Syllogism to get $p \rightarrow r$
$p \vee q$ and $\neg p$	Disjunctive Syllogism to get $q$
$p \wedge q$	Simplification to get $p$ or $q$
$p$ and $q$ separately	Conjunction to get $p \wedge q$
$\forall x P(x)$	Universal Instantiation to get $P(c)$
$\exists x P(x)$	Existential Instantiation to name a witness $c$