Relations

Definition

A **relation** R on a set A is a subset of $A \times A$ (the set of all ordered pairs from A).
If $(a, b) \in R$, we write $a \; R \; b$ and say "a is related to b."
Example: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 3), (3, 3)\}$.
Then $1 \; R \; 1$, $1 \; R \; 2$, $2 \; R \; 3$, and $3 \; R \; 3$, but $2 \not R \; 1$ and $1 \not R \; 3$.

Properties of relations

Property	Definition	In words
Reflexive	$\forall a \in A$: $a \; R \; a$	Every element is related to itself
Irreflexive	$\forall a \in A$: $a \not R \; a$	No element is related to itself
Symmetric	$\forall a, b \in A$: $a \; R \; b \Rightarrow b \; R \; a$	If a relates to b, then b relates to a
Antisymmetric	$\forall a, b \in A$: $(a \; R \; b \wedge b \; R \; a) \Rightarrow a = b$	Mutual relation only when equal
Transitive	$\forall a, b, c \in A$: $(a \; R \; b \wedge b \; R \; c) \Rightarrow a \; R \; c$	Relations "chain" together

Checking properties

Example: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$.
Reflexive? Check if $(a, a) \in R$ for all $a \in A$.
$(1, 1), (2, 2), (3, 3), (4, 4) \in R$ ✓ **Yes, reflexive.**
Symmetric? Check if $(a, b) \in R \Rightarrow (b, a) \in R$.
$(1, 2) \in R$ and $(2, 1) \in R$ ✓ All pairs check out. **Yes, symmetric.**
Antisymmetric? Check if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.
$(1, 2) \in R$ and $(2, 1) \in R$, but $1 \neq 2$ X **No, not antisymmetric.**
Transitive? Check if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.
$(1, 2) \in R$ and $(2, 1) \in R$, so need $(1, 1) \in R$ ✓
$(2, 1) \in R$ and $(1, 2) \in R$, so need $(2, 2) \in R$ ✓
All chains check out. **Yes, transitive.**

Visualizing relations

Relations can be represented as **directed graphs** (digraphs):

- Vertices represent elements of A

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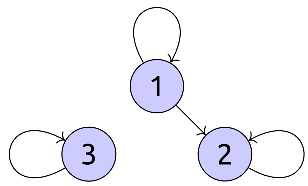
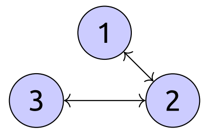
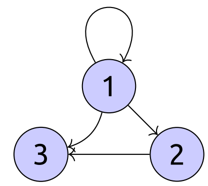
- A directed edge from a to b means $a \; R \; b$
- A loop at a means $a \; R \; a$



Reflexive
(loops at every vertex)



Symmetric
(all edges bidirectional)



Antisymmetric
(no bidirectional edges)

Equivalence relations

An **equivalence relation** is a relation that is:

1. Reflexive
2. Symmetric
3. Transitive

Equivalence relations partition a set into disjoint subsets called **equivalence classes**.
The equivalence class of a is: $[a] = \{x \in A : x \; R \; a\}$
Example: Let R be the relation on $\mathbb{Z}$ defined by $a \; R \; b$ if $a \equiv b \pmod 3$.
Reflexive: $a \equiv a \pmod 3$ ✓
Symmetric: If $a \equiv b \pmod 3$, then $b \equiv a \pmod 3$ ✓
Transitive: If $a \equiv b \pmod 3$ and $b \equiv c \pmod 3$, then $a \equiv c \pmod 3$ ✓
This is an equivalence relation with three equivalence classes:

$$[0] = \{\ldots, -6, -3, 0, 3, 6, 9, \ldots\}$$
$$[1] = \{\ldots, -5, -2, 1, 4, 7, 10, \ldots\}$$
$$[2] = \{\ldots, -4, -1, 2, 5, 8, 11, \ldots\}$$

Note: Symmetric and antisymmetric are **not** opposites! A relation can be both (e.g., equality), neither, or one but not the other.