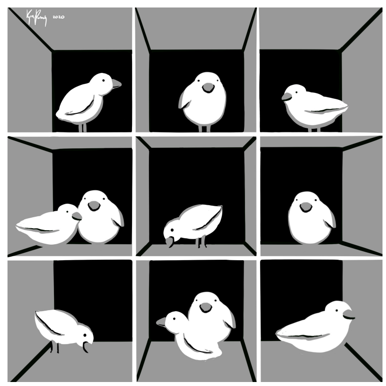
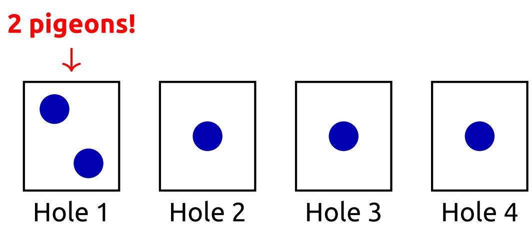
# Pigeonhole principle

## The basic pigeonhole principle

If $n+1$ objects are distributed among $n$ containers, then at least one container must hold **at least 2** objects.



The name comes from a visual image: if 5 pigeons fly into 4 holes, at least one hole must contain at least 2 pigeons.



5 pigeons, 4 holes $\Rightarrow$ at least one hole has $\geq 2$ pigeons

Note: The principle guarantees *existence*—it tells you *that* some container has multiple objects, not *which* one.

## The generalized pigeonhole principle

If $n$ objects are distributed among $k$ containers, then at least one container holds at least $\left\lceil \frac{n}{k} \right\rceil$ objects.

Here $\lceil \cdot \rceil$ is the **ceiling function**: round up to the nearest integer.

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| Objects ($n$) | Containers ($k$) | $\lceil n/k \rceil$ | Conclusion |
|---|---|---|---|
| 13 | 12 | $\lceil 13/12 \rceil = 2$ | At least 2 objects share a container |
| 25 | 12 | $\lceil 25/12 \rceil = 3$ | At least 3 objects share a container |
| 100 | 7 | $\lceil 100/7 \rceil = 15$ | At least 15 objects share a container |

Tip: To find the **minimum** $n$ that guarantees at least $m$ objects in one container, solve $\lceil n/k \rceil \geq m$, which gives $n \geq k(m-1)+1$.

## The key skill: identifying pigeons and pigeonholes

Most pigeonhole problems hide the pigeons and holes inside a word problem. Use this strategy:

| Ask yourself... | Pigeons | Pigeonholes |
|---|---|---|
| Birthdays | People | Days or months of the year |
| Remainders | Integers | Possible remainders ($0, 1, \dots, k-1$) |
| Sums/differences | Pairs of numbers | Possible values of the sum/difference |
| Grid problems | Points placed | Cells of the grid |
| Last digits | Integers | Digits $0$ through $9$ |

The **number of pigeonholes** is typically determined by a limited set of categories (months, remainders, grid cells, etc.). Once you know $n$ (pigeons) and $k$ (holes), apply the formula.

## Examples

Example 1 (Basic): In any group of 13 people, at least 2 must share a birth month.
**Solution:** There are 12 months, so $k = 12$ (holes). With 13 people ($n = 13$ pigeons):

$$\left\lceil \frac{13}{12} \right\rceil = 2$$

At least one month is shared by at least 2 people. □

Example 2 (Minimum count): What is the minimum number of people needed to guarantee that at least **3** share a birth month?
**Solution:** We need $\lceil n/12 \rceil \geq 3$, so we need $n \geq 12(3-1)+1 =$ **25** people.
*Why not 24?* With 24 people, it is possible (though unlikely) that exactly 2 are born each month—no month has 3. The 25th person must join some month already holding 2. □

Example 3 (Remainders): Among any 5 integers, at least 2 must have the same remainder when divided by 4.

**Solution:** When dividing by 4, the only possible remainders are $0, 1, 2, 3$—so $k = 4$ holes. With 5 integers as pigeons:

$$\left\lceil \frac{5}{4} \right\rceil = 2$$

At least 2 integers land in the same remainder class. □

Example 4 (Socks in a drawer): A drawer contains 6 red socks and 6 blue socks in the dark. How many socks must you draw to **guarantee** a matching pair?

**Solution:** There are 2 colours (holes). Drawing 3 socks (pigeons):

$$\left\lceil \frac{3}{2} \right\rceil = 2$$

At least 2 socks share the same colour, giving a matching pair. The answer is **3**.

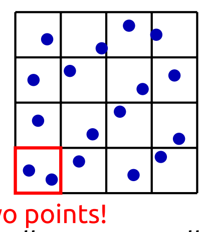
Caution: The total of 12 socks is irrelevant—colour is the only attribute that matters here. □

Example 5 (Grid): A 4×4 grid of unit squares has 17 points placed inside it (no point on a grid line). Show that at least 2 points lie in the same unit square.

**Solution:** There are $4 \times 4 = 16$ unit squares (holes) and 17 points (pigeons):

$$\left\lceil \frac{17}{16} \right\rceil = 2$$

At least 2 points must share the same unit square. □




17 points, 16 cells $\Rightarrow$ some cell has $\geq 2$ points

## Proof structure

Every pigeonhole proof has the same three sentences:

1. **Name the pigeons.** "There are $n$ objects (pigeons)…"
2. **Name the pigeonholes.** "…and only $k$ categories (pigeonholes)…"
3. **Apply the principle.** "By the (generalized) pigeonhole principle, at least one category contains $\lceil n/k \rceil$ objects."

## Common mistakes

Caution:

- **Confusing "guarantee" with "likely."** The principle gives a *worst-case* guarantee. Never say "it is possible that…" when asked to prove something must happen.
- **Off-by-one errors in minimum counts.** To guarantee $m$ per hole, you need $k(m-1)+1$ pigeons, *not* $km$. With $km$ pigeons, it is still possible to have exactly $m$ in every hole and none with $m+1$.
- **Wrong pigeonhole count.** Count categories carefully. Remainders mod 4 give *four* holes ($0, 1, 2, 3$), not three.
- **Forgetting the ceiling.** $\lceil 7/3 \rceil = 3$, not 2. Always round *up*.

## Quick reference

| Situation | Formula |
|---|---|
| Basic: guarantee 2 share a hole | Need $n \geq k+1$ pigeons |
| Generalized: guarantee $m$ share a hole | Need $n \geq k(m-1)+1$ pigeons |
| Given $n$ pigeons and $k$ holes | At least $\lceil n/k \rceil$ in one hole |

**Practice problems.** For each problem, identify the pigeons and pigeonholes, then apply the principle.

**1.** How many people must be in a room to guarantee that at least 2 were born on the same day of the week?

**2.** How many cards must be drawn from a standard 52-card deck to guarantee at least 2 cards share the same suit?

**3.** Among any 11 integers, show that at least 2 must have the same remainder when divided by 10.

**4.** A bag contains socks in 5 different colours. How many socks must you draw to guarantee a matching pair?

**5.** What is the minimum number of students needed to guarantee that at least 5 share a birth month?

**6.** A class has 30 students who each scored an integer between 0 and 100 on a test. Show that at least 2 students received the same grade, or explain why this cannot be guaranteed.

**7.** Show that among any 11 integers, at least two must have the same last digit.

**8.** How many points must be placed inside a $3 \times 5$ rectangle (divided into unit squares) to guarantee 2 points share a unit square?

**Answers:** 1. 8 people (7 days + 1)  2. 5 cards (4 suits + 1)  3. 11 integers, 10 remainders ($0$–$9$); $\lceil 11/10 \rceil = 2$  4. 6 socks ($5 + 1$)

5. $12(5-1)+1 = 49$ students  6. Cannot be guaranteed: 101 possible grades, only 30 students, so all 30 could be distinct  7. 11 integers, 10 possible last digits ($0$–$9$); $\lceil 11/10 \rceil = 2$  8. $3 \times 5 = 15$ squares, so 16 points