Logarithms

What is a logarithm?

A logarithm is the **inverse operation of exponentiation**. The two forms below say the same thing:

Exponential form	$\longleftrightarrow$	**Logarithmic form**
$b^x = y$	$\longleftrightarrow$	$\log_b y = x$

We read $\log_b y = x$ as *"the logarithm base b of y equals x."* In other words: $\log_b y$ asks *"what power must b be raised to in order to get y?"*

Example: Since $2^5 = 32$, we have $\log_2 32 = 5$.

Example: Since $10^3 = 1000$, we have $\log_{10} 1000 = 3$.

Example: Since $5^0 = 1$, we have $\log_5 1 = 0$.

Note: Logarithms are only defined for **positive** inputs: $\log_b x$ requires $x > 0$ and $b > 0$, $b \neq 1$.

Common logarithms

Two bases are used so often they have dedicated notation:

Name	Base	Notation
Common logarithm	$b = 10$	$\log x$ (base 10 is implied)
Natural logarithm	$b = e \approx 2.71828$	$\ln x$

Example: Convert between logarithmic and exponential forms.

(a) $\log_3 81 = 4 \iff 3^4 = 81$
(b) $\ln y = 2 \iff e^2 = y$
(c) $\log 1000 = 3 \iff 10^3 = 1000$

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Properties of logarithms

Property	Example
$\log_b 1 = 0$	$\log_7 1 = 0$
$\log_b b = 1$	$\log_5 5 = 1$
$\log_b b^x = x$	$\log_3 3^7 = 7$
$b^{\log_b x} = x$	$10^{\log 4} = 4$
Product rule: $\log_b(xy) = \log_b x + \log_b y$	$\log_2(8 \cdot 4) = \log_2 8 + \log_2 4 = 3 + 2 = 5$
Quotient rule: $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log\left(\frac{1000}{10}\right) = \log 1000 - \log 10 = 3 - 1 = 2$
Power rule: $\log_b(x^n) = n \log_b x$	$\ln\left(e^{3t}\right) = 3t \ln e = 3t$
Change of base: $\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}$	$\log_2 10 = \frac{\ln 10}{\ln 2} \approx \frac{2.303}{0.693} \approx 3.322$

Caution: $\log_b(x + y) \neq \log_b x + \log_b y$. The product rule applies to log of a *product*, not a sum.

Expanding and condensing logarithms

The log rules can be used in two directions:
Expanding breaks a single log into simpler parts. **Condensing** combines multiple logs into one.
Example: Expand $\log\left(\frac{x^3 y}{z^2}\right)$.

$$\begin{aligned} \log\left(\frac{x^3 y}{z^2}\right) &= \log\left(x^3 y\right) - \log\left(z^2\right) \\ &= \log\left(x^3\right) + \log y - \log\left(z^2\right) \\ &= 3 \log x + \log y - 2 \log z \end{aligned}$$

Example: Condense $2 \ln x - \ln(x + 1) + 3 \ln y$ into a single logarithm.

$$\begin{aligned} 2 \ln x - \ln(x + 1) + 3 \ln y &= \ln\left(x^2\right) + \ln\left(y^3\right) - \ln(x + 1) \\ &= \ln\left(\frac{x^2 y^3}{x + 1}\right) \end{aligned}$$

Solving equations using logarithms

To solve for a variable in an exponent, isolate the exponential expression, then apply ln to both sides and use the power rule.
Example: Solve $3^x = 20$.

Taking ln of both sides:

$$\ln(3^x) = \ln 20 \implies x \ln 3 = \ln 20 \implies x = \frac{\ln 20}{\ln 3} \approx \frac{2.996}{1.099} \approx 2.727$$

Example: Solve $5e^{2x} = 40$.

$$e^{2x} = 8 \implies \ln\left(e^{2x}\right) = \ln 8 \implies 2x = \ln 8 \implies x = \frac{\ln 8}{2} \approx 1.040$$

Practice problems.

Convert between forms.

1. Write $4^3 = 64$ in log form. **2.** Write $\log_6 216 = 3$ in exponential form. **3.** Write $e^2 = y$ in log form.

Expand each logarithm.

5. $\log(x^2y^3)$ **6.** $\ln\left(\frac{\sqrt{x}}{y^4}\right)$ **7.** $\log_2\left(\frac{8x^5}{z}\right)$

Condense into a single logarithm.

8. $3\log x + \log y$ **9.** $\ln a - 2\ln b + \ln c$

10. $2\log_3 x - \log_3(x+1)$ **11.** $\frac{1}{2}\ln x + 3\ln y - \ln z$

Solve for x. Round to three decimal places where needed.

12. $2^x = 50$ **13.** $e^{3x} = 12$ **14.** $4e^x - 1 = 19$ **15.** $7^{2x+1} = 100$

Answers: 1. $\log_4 64 = 3$ 2. $6^3 = 216$ 3. $\ln y = 2$ 4. 3 5. $2\log x + 3\log y$ 6. $\frac{1}{2}\ln x - 4\ln y$ 7. $3 + 5\log_2 x - \log_2 z$ 8. $\log(x^3y)$ 9. $\ln\left(\frac{ac}{b^2}\right)$ 10. $\log_3\left(\frac{x^2}{x+1}\right)$ 11. $\ln\left(\frac{\sqrt{x}\,y^3}{z}\right)$ 12. $x \approx 5.644$ 13. $x \approx 0.828$ 14. $x \approx 1.609$ 15. $x \approx 0.683$