Linear programming

What is linear programming?

Linear programming (LP) finds the **maximum or minimum** of an objective function subject to a set of linear constraints. It is used whenever resources are limited and you want the best possible outcome.

Key vocabulary:

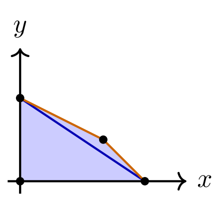
Term	**Meaning**
Objective function	The quantity to maximize or minimize (e.g. profit, cost)
Constraints	Linear inequalities restricting the variables
Feasible region	All points satisfying *every* constraint simultaneously
Corner point	A vertex of the feasible region

The fundamental theorem of linear programming

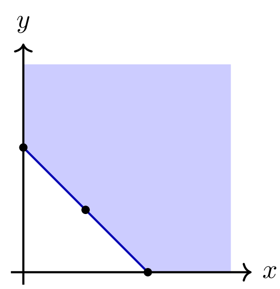
> **If an optimal solution exists, it occurs at a *corner point* of the feasible region.**
>
> To find the optimum: evaluate the objective function at *every* corner point and pick the best.

Note: If the feasible region is **unbounded**, a maximum may not exist. A minimum still exists if all objective function coefficients are positive.

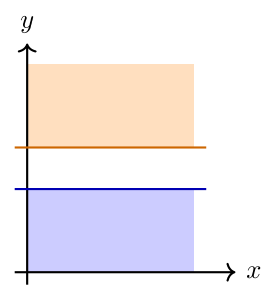
Solution region types



Bounded
Finite feasible region
Max & min both exist



Unbounded
Region extends to infinity
Min may exist; max may not



No solution
Constraints don't overlap
No feasible region exists

Step-by-step method

1. **Define variables.** Let $x = \ldots$ and $y = \ldots$ (name the quantities you control).

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2. **Write the objective function.** State what to maximize or minimize:

   Maximize (or minimize): $P = ax + by$

3. **Write the constraints.** Translate each restriction into a linear inequality. Include implicit constraints such as $x \geq 0$, $y \geq 0$ (quantities cannot be negative).

4. **Graph the feasible region.**

   - Graph each constraint line (use intercepts).
   - Shade each individual half-plane, then identify the *overlap*.
   - Use a dashed line for strict inequalities ($<$, $>$); solid line for $\leq$, $\geq$.

5. **Find all corner points.** Read coordinates from the graph, or solve the system of two constraint equations at each intersection algebraically.

6. **Evaluate and choose.** Substitute each corner point into the objective function. The largest (maximizing) or smallest (minimizing) value is the answer.

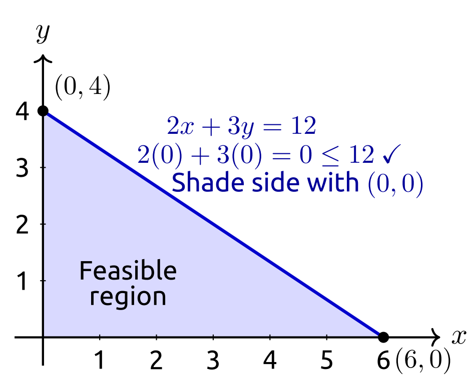
Tip: Always check *every* corner point—you cannot tell which is optimal just by looking at the graph.

Graphing a constraint: quick method

To graph $ax + by \leq c$ (or $\geq c$):

1. Set $y = 0$ to get the x-intercept: $x = c/a$.
2. Set $x = 0$ to get the y-intercept: $y = c/b$.
3. Draw the line through the two intercepts.
4. Test the origin $(0, 0)$: if it satisfies the inequality, shade the side containing $(0, 0)$; otherwise shade the opposite side.

Example: Graph $2x + 3y \leq 12$.



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Finding corner points

Corner points occur where two constraint boundaries **intersect**. Solve the two corresponding equations simultaneously.

Example: Find the intersection of $x + y = 6$ and $x + 2y = 8$.

Solution: Subtract the first equation from the second:

$$(x + 2y) - (x + y) = 8 - 6 \longrightarrow y = 2$$

Substitute back: $x + 2 = 6 \longrightarrow x = 4$. Corner point: $(4, 2)$.

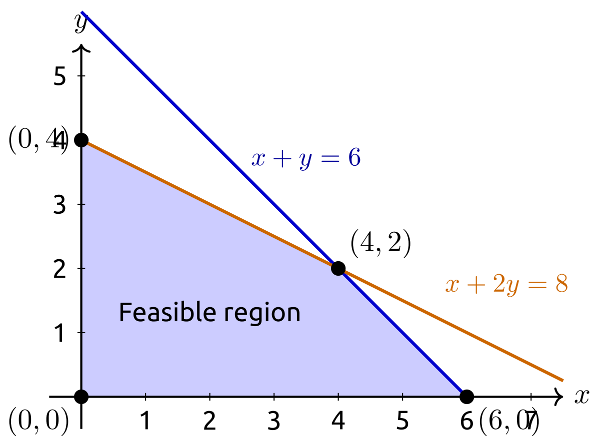
Note: Also include the axis intercepts as corner points (where a constraint line meets $x = 0$ or $y = 0$), and the origin $(0, 0)$ if it is in the feasible region.

Full example: maximization

Example: Maximize $P = 60x + 20y$ subject to:

$$x + y \leq 6, \quad x + 2y \leq 8, \quad x \geq 0, \quad y \geq 0$$

Step 1 – Graph the feasible region.



Step 2 – Identify corner points.

The four corners of the feasible region are $(0, 0)$, $(6, 0)$, $(4, 2)$, and $(0, 4)$.

The point $(4, 2)$ is found by solving $x + y = 6$ and $x + 2y = 8$ simultaneously (see above).

Step 3 – Evaluate $P = 60x + 20y$ at each corner.

Corner	x	y	$P = 60x + 20y$
$(0, 0)$	0	0	$60(0) + 20(0) = 0$
$(6, 0)$	6	0	$60(6) + 20(0) = 360$
$(4, 2)$	4	2	$60(4) + 20(2) = 280$
$(0, 4)$	0	4	$60(0) + 20(4) = 80$

The maximum value is $P = 360$, achieved at $(6, 0)$.

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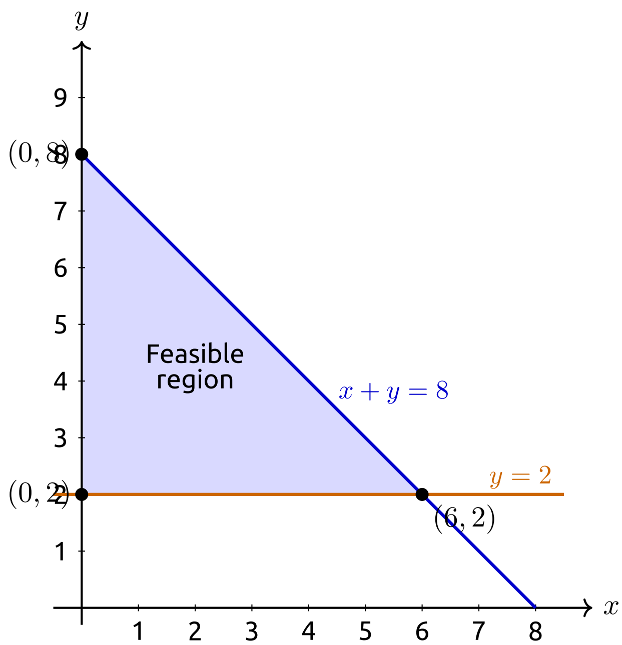
Full example: minimization

Example: A company can produce a **basic** item (profit \$40) and a **premium** item (profit \$60). Assembly capacity limits total units to at most 8, and demand requires at least 2 premium items. How many of each should be made to **minimize cost**, given cost is $C = 20x + 50y$?
Constraints:

$$x + y \leq 8, \quad y \geq 2, \quad x \geq 0, \quad y \geq 0$$

Corner points:



Corner points: $(0, 2)$, $(6, 2)$, $(0, 8)$.

Note: $(6, 2)$ is found by substituting $y = 2$ into $x + y = 8$, giving $x = 6$.
Evaluate $C = 20x + 50y$**:**

Corner	$C = 20x + 50y$
$(0, 2)$	$20(0) + 50(2) = 100$
$(6, 2)$	$20(6) + 50(2) = 220$
$(0, 8)$	$20(0) + 50(8) = 400$

The minimum cost is $C = \$100$ **at** $(0, 2)$**:** produce 0 basic and 2 premium items.

Common mistakes

Caution:

- **Missing a corner point.** Systematically solve every pair of boundary lines to find all intersections – then check each is actually in the feasible region.

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- **Forgetting non-negativity constraints.** Always include $x \geq 0$ and $y \geq 0$ unless the problem says otherwise.
- **Testing interior points instead of corners.** The optimal value is always at a corner, never in the interior.
- **Wrong shading.** Test the origin after drawing each line to confirm which side to shade.

Summary: the LP checklist

- ☐ Define x and y clearly
- ☐ Write objective function: Maximize/Minimize $P = ax + by$
- ☐ List all constraints (include $x \geq 0$, $y \geq 0$)
- ☐ Find intercepts for each line; graph and shade correctly
- ☐ Identify the feasible region (overlap of all shaded areas)
- ☐ Find *all* corner points (axis intercepts + line intersections)
- ☐ Evaluate P at every corner; choose max or min
- ☐ State the answer in context

Practice problems. For each problem, define variables, write the objective function and constraints, graph the feasible region, find all corner points, and determine the optimal value.

1. Maximize $P = 5x + 4y$ subject to $x + y \leq 5$, $2x + y \leq 8$, $x \geq 0$, $y \geq 0$.

2. Maximize $P = 3x + 5y$ subject to $x + 2y \leq 10$, $3x + y \leq 12$, $x \geq 0$, $y \geq 0$.

3. Minimize $C = 6x + 10y$ subject to $x + y \geq 6$, $2x + y \geq 8$, $x \geq 0$, $y \geq 0$.

4. Minimize $C = 4x + 7y$ subject to $x + 2y \geq 8$, $x + y \geq 5$, $x \geq 0$, $y \geq 0$.

5. A company makes chairs (profit \$30) and tables (profit \$70). Each chair takes 2 hrs and each table takes 5 hrs; 40 hrs of labour are available. At most 12 chairs can be made per week. Maximize profit.

6. A dietitian needs at least 24 units of vitamin A and 16 units of vitamin B. Food X provides 4 units of A and 2 units of B per serving at \$3. Food Y provides 2 units of A and 4 units of B per serving at \$4. Minimize cost.

Answers: 1. $P = 23$ at $(3, 2)$ 2. $P = 25$ at $(2, 4)$ 3. $C = 32$ at $(2, 4)$ 4. $C = 20$ at $(5, 0)$
5. Max profit $= \$620$ at $(5, 6)$: 5 tables and 12 chairs 6. Min cost $= \$28$ at $(4, 4)$: 4 servings of each

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