Linear approximation

What is linear approximation?

Linear approximation (or **linearization**) uses the tangent line to approximate a function near a point.
If x is close to a, then

$$f(x) \approx L(x),$$

where $L(x)$ is the tangent line at $x = a$.
The tangent line is easier to work with than the original function, so we use it to estimate nearby values.

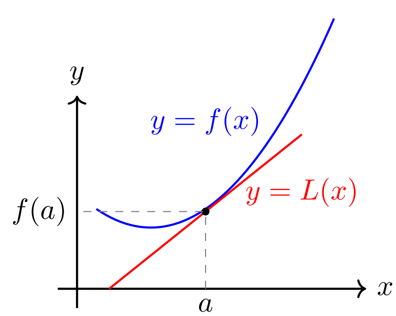
The linearization formula

For a differentiable function f, the linearization at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a).$$

Idea:

- $f(a)$ gives the starting value
- $f'(a)$ gives the slope of the tangent line
- $(x - a)$ measures how far we moved from a



Finding tangent lines using linearization

The linearization formula

$$L(x) = f(a) + f'(a)(x - a)$$

is also the equation of the tangent line to the curve at $x = a$.

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Example: Find the equation of the tangent line to $f(x) = \sqrt{x+5}$ at $x = 4$.

Solution:
First find the derivative:

$$f'(x) = \frac{1}{2\sqrt{x+5}}.$$

Now evaluate the function and derivative at $x = 4$:

$$f(4) = \sqrt{9} = 3, \qquad f'(4) = \frac{1}{2\sqrt{9}} = \frac{1}{6}.$$

Use the linearization formula:

$$L(x) = f(a) + f'(a)(x - a).$$

Substitute $a = 4$:

$$L(x) = 3 + \frac{1}{6}(x - 4).$$

This is the tangent line. It can also be written as

$$y = \frac{1}{6}x + \frac{7}{3}.$$

How to find a linear approximation

Step 1: Choose a nearby value a where the function is easy to evaluate.
Step 2: Find $f(a)$ and $f'(a)$.
Step 3: Substitute into

$$L(x) = f(a) + f'(a)(x - a).$$

Step 4: Use $L(x)$ to approximate $f(x)$.
Tip: Choose a value of a close to x where the function and derivative are easy to compute.
Caution: Linear approximation becomes less accurate as x moves farther away from a.

Approximating values

Example: Approximate $\sqrt{4.1}$.

Solution:
Let

$$f(x) = \sqrt{x}.$$

Then

$$f'(x) = \frac{1}{2\sqrt{x}}.$$

Choose $a = 4$ since $\sqrt{4} = 2$ is easy to compute.

Compute:

$$f(4) = 2, \qquad f'(4) = \frac{1}{4}.$$

The linearization is

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

Now substitute $x = 4.1$:

$$L(4.1) = 2 + \frac{1}{4}(0.1) = 2.025.$$

Therefore,

$$\sqrt{4.1} \approx \boxed{2.025}.$$

Actual value:

$$\sqrt{4.1} \approx 2.0248.$$

Error: ≈ 0.0002.

Example: Approximate $\sin(0.1)$.

Solution:
Let $f(x) = \sin x$. Then $f'(x) = \cos x$.
Choose $a = 0$ since $\sin(0) = 0 \quad \text{and} \quad \cos(0) = 1$.
Compute:

$$f(0) = 0, \qquad f'(0) = 1.$$

The linearization is $L(x) = 0 + 1(x - 0) = x$.
Substitute $x = 0.1$:

$$L(0.1) = 0.1.$$

Therefore,

$$\sin(0.1) \approx \boxed{0.1}.$$

Actual value:

$$\sin(0.1) \approx 0.0998.$$

Summary

Step	What to do
Choose a	Pick a nearby value where the function is easy to evaluate.
Find $f(a)$	Compute the function value at a.
Find $f'(a)$	Compute the slope of the tangent line at a.
Build $L(x)$	Use $L(x) = f(a) + f'(a)(x - a)$ to create the approximation.
Approximate $f(x)$	Substitute the desired value of x into $L(x)$.

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Practice problems

Find the equation of the tangent line using linearization:

1. $f(x) = x^3$ at $x = 2$

2. $f(x) = \sqrt{x+1}$ at $x = 3$

3. $f(x) = \dfrac{1}{x}$ at $x = 2$

4. $f(x) = \sin x$ at $x = 0$

Use linear approximation to estimate each value.

5. $\sqrt{9.1}$

6. $\sqrt[3]{26.9}$

7. $\sin(0.05)$

8. $\dfrac{1}{1.1}$

9. $\dfrac{1}{\sqrt{3.8}}$

10. $\sqrt{15.8}$

11. $(1.02)^5$

12. $\ln(1.1)$

13. $\sqrt{48.2}$

14. $e^{0.08}$

Answers

1. $y = 12x - 16$ 2. $y = \frac{1}{4}x + \frac{5}{4}$ 3. $y = -\frac{1}{4}x + 1$ 4. $y = x$

5. 3.0167 6. 2.9963 7. 0.05 8. 0.9 9. 0.5125 10. 3.975 11. 1.10 12. 0.1

13. 6.9429 14. 1.08

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