Limits by the Squeeze Theorem

The squeeze theorem

The squeeze theorem is useful when one function is trapped between two simpler functions. If

$$g(x) \leq f(x) \leq h(x)$$

near $x = a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} f(x) = L.$$

Idea: If $f(x)$ is squeezed between two functions that both approach the same value, then $f(x)$ must approach that value too.

How to use the squeeze theorem

1. Find a lower bound and an upper bound for the function.
2. Check that both bounds approach the same value.
3. Conclude that the middle function has the same limit.

Note: The squeeze theorem is often used for sine and cosine, but it can be used whenever a function is trapped between two simpler functions with the same limit.

A common trig fact

A common use of the squeeze theorem comes from the facts

$$-1 \leq \sin(\text{anything}) \leq 1$$

and

$$-1 \leq \cos(\text{anything}) \leq 1.$$

This is true no matter what is inside the sine or cosine.
For example,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

even though $\frac{1}{x}$ is undefined at $x = 0$.

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A trig limit

Example: Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution:

Start with

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1.$$

Since $x^2 \ge 0$, multiply each part by x^2:

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2.$$

Since

$$\lim_{x\to 0}(-x^2) = 0 \qquad \text{and} \qquad \lim_{x\to 0} x^2 = 0,$$

the squeeze theorem gives

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \boxed{0}.$$

Even powers of sine and cosine

Example: Evaluate $\lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x}\right)$.

Solution:

Since cosine is always between -1 and 1,

$$-1 \le \cos\left(\frac{1}{x}\right) \le 1.$$

Squaring makes the expression non-negative, so

$$0 \le \cos^2\left(\frac{1}{x}\right) \le 1.$$

Now multiply each part by x^2:

$$0 \le x^2 \cos^2\left(\frac{1}{x}\right) \le x^2.$$

Since both outside functions approach 0,

$$\lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x}\right) = \boxed{0}.$$

Example: Evaluate $\lim_{x\to 0}\left(4 + x^2 \cos\left(\frac{3}{x}\right)\right)$.

Solution:

Since

$$-1 \leq \cos\left(\frac{3}{x}\right) \leq 1,$$

we get

$$-x^2 \leq x^2 \cos\left(\frac{3}{x}\right) \leq x^2.$$

Add 4 to each part:

$$4 - x^2 \leq 4 + x^2 \cos\left(\frac{3}{x}\right) \leq 4 + x^2.$$

Both outside functions approach 4, so

$$\lim_{x\to 0}\left(4 + x^2 \cos\left(\frac{3}{x}\right)\right) = \boxed{4}.$$

Given inequalities

Example: Suppose $7 - (x-2)^2 \leq f(x) \leq 7 + |x-2|$.
Find

$$\lim_{x\to 2} f(x).$$

Solution:
Look at the lower bound:

$$\lim_{x\to 2}\left(7 - (x-2)^2\right) = 7.$$

Look at the upper bound:

$$\lim_{x\to 2}\left(7 + |x-2|\right) = 7.$$

Since $f(x)$ is trapped between two functions that both approach 7,

$$\lim_{x\to 2} f(x) = \boxed{7}.$$

Example: Suppose $-1 \leq f(x) \leq 5$.
Find

$$\lim_{x\to\infty} \frac{f(x)}{e^x}.$$

Solution: Since $e^x > 0$, divide each part by e^x:

$$-\frac{1}{e^x} \leq \frac{f(x)}{e^x} \leq \frac{5}{e^x}.$$

As $x \to \infty$,

$$-\frac{1}{e^x} \to 0 \qquad \text{and} \qquad \frac{5}{e^x} \to 0.$$

Therefore,

$$\lim_{x\to\infty} \frac{f(x)}{e^x} = \boxed{0}.$$

Summary

Situation	What to do
A function is trapped between two simpler functions	Check whether the outside functions have the same limit.
The function involves sin or cos	Use the fact that sine and cosine are always between -1 and 1.
The outside functions both approach L	The middle function must also approach L.

Caution: If you multiply or divide an inequality by a negative quantity, the inequality signs reverse.

Practice problems. Evaluate each limit.

1. $\lim_{x\to 0} x^3 \cos\left(\frac{5}{x^2}\right)$

2. $\lim_{x\to 0} \sqrt{x^2+1}\cdot x^2 \sin\left(\frac{1}{x}\right)$

3. $\lim_{x\to 0} x^2 \left(3+\sin^2\left(\frac{2}{x}\right)\right)$

4. $\lim_{x\to 0^+} \sqrt{x}\cos^4\left(\frac{1}{\sqrt{x}}\right)$

5. $\lim_{x\to 0} \left(x^2 \cos\left(\frac{1}{x}\right)-2\right)$

6. $\lim_{x\to 0} \left(5-x^4 \sin\left(\frac{\pi}{x}\right)\right)$

7. Suppose $\frac{2x}{\sin(x)} \le f(x) \le 2-x^2$ near $x=0$. Find $\lim_{x\to 0} f(x)$.

8. Suppose $1-\ln(x+1) \le f(x) \le e^x$. Find $\lim_{x\to 0} f(x)$.

9. Suppose $4-|x-3| \le f(x) \le \frac{x^2+3}{3}$. Find $\lim_{x\to 3} f(x)$.

10. Suppose $-2 \le f(x) \le 7$. Find $\lim_{x\to\infty} \frac{f(x)}{x^2+1}$.

11. Suppose $0 \le f(x) \le \tan^{-1}(x)$. Find $\lim_{x\to 0} f(x)$.

12. Suppose $-5 \le f(x) \le 5$. Find $\lim_{x\to\infty} \left(2+\frac{f(x)}{3^x}\right)$.

Answers: 1. 0 2. 0 3. 0 4. 0 5. -2 6. 5 7. 2 8. 1 9. 4 10. 0 11. 0 12. 2