Limits by rationalizing

When to rationalize

Use rationalizing when direct substitution gives $\frac{0}{0}$ and the expression contains a **square root** that prevents factoring directly. The idea is to eliminate the radical by multiplying by a **conjugate**, which creates a difference of squares in its place.

Conjugate pairs:

Expression	Conjugate
$\sqrt{f(x)} - a$	$\sqrt{f(x)} + a$
$\sqrt{f(x)} + a$	$\sqrt{f(x)} - a$
$\sqrt{f(x)} - \sqrt{g(x)}$	$\sqrt{f(x)} + \sqrt{g(x)}$

Multiplying by $\frac{\text{conjugate}}{\text{conjugate}}$ equals 1, so the limit is unchanged.
Key identity used: $(a-b)(a+b) = a^2 - b^2$, which removes the radical.

Rationalizing the numerator

This is the most common case: a radical appears in the **numerator** and produces $\frac{0}{0}$.

Example: Evaluate $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$.

Solution: Direct substitution gives $\frac{\sqrt{4}-2}{0} = \frac{0}{0}$. Multiply numerator and denominator by the conjugate $\sqrt{x+4}+2$:

$$\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \lim_{x\to 0} \frac{(\sqrt{x+4})^2 - 2^2}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{x+4-4}{x(\sqrt{x+4}+2)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4} = \boxed{\frac{1}{4}}$$

Example: Evaluate $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$.

Solution: Direct substitution gives $\frac{0}{0}$. Multiply by the conjugate $\sqrt{x+1}+2$:

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4} = \boxed{\frac{1}{4}}$$

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Tip: After multiplying by the conjugate, look for a factor in the numerator that cancels with the denominator. If no cancellation occurs, re-check your algebra: it should always cancel when the form was genuinely $\frac{0}{0}$.

Rationalizing when a is not zero

Example: Evaluate $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$.

Solution: Direct substitution gives $\dfrac{0}{0}$. Here the radical is in the **denominator**. Multiply by the conjugate $\sqrt{x}+3$:

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9} = \lim_{x\to 9}(\sqrt{x}+3) = \sqrt{9}+3 = \boxed{6}$$

Note: When the radical is in the denominator, multiplying by the conjugate moves the cancellable factor to the numerator instead. The approach is identical.

Rationalizing with two radicals

When both terms in a difference are radicals, multiply by the conjugate of the entire expression.

Example: Evaluate $\lim_{x\to 4} \dfrac{\sqrt{2x+1}-3}{\sqrt{x}-2}$.

Solution: Direct substitution gives $\dfrac{0}{0}$. Rationalize the **numerator** first:

$$\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{\sqrt{x}-2} \cdot \frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3} = \lim_{x\to 4} \frac{(2x+1)-9}{(\sqrt{x}-2)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{2x+1}+3)}$$

Now rationalize the remaining $(\sqrt{x}-2)$ by multiplying by $(\sqrt{x}+2)$:

$$= \lim_{x\to 4} \frac{2(x-4)(\sqrt{x}+2)}{(x-4)(\sqrt{2x+1}+3)} = \lim_{x\to 4} \frac{2(\sqrt{x}+2)}{\sqrt{2x+1}+3} = \frac{2(\sqrt{4}+2)}{\sqrt{9}+3} = \frac{2(4)}{6} = \frac{8}{6} = \boxed{\frac{4}{3}}$$

Combining rationalizing with factoring

Sometimes rationalizing alone isn't enough and the remaining expression must also be factored.

Example: Evaluate $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x^2+3}-2}$.

Solution: Direct substitution gives $\dfrac{0}{0}$. Rationalize the denominator:

$$\lim_{x\to 1} \frac{x-1}{\sqrt{x^2+3}-2} \cdot \frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x^2+3)-4} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{x^2-1}$$

Factor the denominator: $x^2 - 1 = (x-1)(x+1)$:

$$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x^2+3}+2)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{\sqrt{x^2+3}+2}{x+1} = \frac{\sqrt{4}+2}{2} = \frac{4}{2} = \boxed{2}$$

Summary: procedure for limits by rationalizing

Step 1 Try direct substitution. If you get $\frac{0}{0}$ and a radical is present, rationalize.

Step 2 Multiply numerator and denominator by the conjugate of the radical expression.

Step 3 Expand using $(a-b)(a+b) = a^2 - b^2$ to eliminate the radical.

Step 4 Cancel the common factor that caused the $\frac{0}{0}$ form.

Step 5 Substitute $x = a$ into the simplified expression.

Caution: Only expand the product that contains the conjugate; leave the other factor **unexpanded**. Expanding everything prematurely makes cancellation much harder to spot.

Practice problems. Evaluate each limit.

1. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$

2. $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+16}-4}{x}$

3. $\displaystyle\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5}$

4. $\displaystyle\lim_{x\to 16} \frac{x-16}{\sqrt{x}-4}$

5. $\displaystyle\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}$

6. $\displaystyle\lim_{x\to 0} \frac{\sqrt{4+x}-\sqrt{4-x}}{x}$

7. $\displaystyle\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3}$

8. $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{3x+1}-2}$

9. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

10. $\displaystyle\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{\sqrt{x+2}-2}$

Answers: 1. $\frac{1}{6}$ 2. $\frac{1}{8}$ 3. $\frac{1}{6}$ 4. 8 5. $\frac{1}{10}$ 6. 1 7. $\frac{3}{4}$ 8. $\frac{2}{3}$ 9. 1 10. $\frac{4}{3}$

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