Limits by factoring

Why factoring is needed

When evaluating $\lim_{x\to a} f(x)$, the first step is always **direct substitution**. If substituting $x = a$ gives a defined value, that value is the limit.
Factoring is needed when direct substitution produces the indeterminate form $\frac{0}{0}$. This means both the numerator and denominator share a **common factor of** $(x-a)$ that can be cancelled, resolving the indeterminate form.
The strategy:

1. Try direct substitution. If you get $\frac{0}{0}$, proceed.
2. Factor the numerator and denominator completely.
3. Cancel the common factor (valid because $x \to a$ means $x \neq a$, so dividing by the factor is legal).
4. Substitute $x = a$ into the simplified expression.

Note: Cancelling the common factor does not change the limit. The original function and the simplified function agree everywhere *except* at $x = a$, and limits don't depend on the value at the point.

Factoring a difference of squares

Recall: $a^2 - b^2 = (a-b)(a+b)$.
Example: Evaluate $\lim_{x\to 3} \frac{x^2-9}{x-3}$.
Solution: Direct substitution gives $\frac{9-9}{3-3} = \frac{0}{0}$. Factor the numerator:

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = \boxed{6}$$

Factoring a trinomial

Example: Evaluate $\lim_{x\to -2} \frac{x^2+5x+6}{x+2}$.
Solution: Direct substitution gives $\frac{4-10+6}{0} = \frac{0}{0}$. Factor the numerator:

$$\lim_{x\to -2} \frac{x^2+5x+6}{x+2} = \lim_{x\to -2} \frac{(x+2)(x+3)}{x+2} = \lim_{x\to -2}(x+3) = -2+3 = \boxed{1}$$

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Factoring in numerator and denominator

Sometimes both the numerator and denominator need to be factored.

Example: Evaluate $\lim_{x \to 4} \dfrac{x^2 - 3x - 4}{x^2 - 16}$.

Solution: Direct substitution gives $\dfrac{16 - 12 - 4}{16 - 16} = \dfrac{0}{0}$. Factor both:

$$\lim_{x \to 4} \frac{x^2 - 3x - 4}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x+1)}{(x-4)(x+4)} = \lim_{x \to 4} \frac{x+1}{x+4} = \frac{5}{8} = \boxed{\frac{5}{8}}$$

Limits from the left and right (one-sided limits)

If a simplified function is still undefined at $x = a$, check whether the limit exists by testing both sides.

Example: Evaluate $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 - 2x + 1}$.

Solution: Direct substitution gives $\dfrac{0}{0}$. Factor:

$$\lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)^2} = \lim_{x \to 1} \frac{x+1}{x-1}$$

After cancellation, $x = 1$ still makes the denominator zero, but the numerator equals $2 \neq 0$. Test each side:

$$\lim_{x \to 1^-} \frac{x+1}{x-1} = \frac{2^-}{0^-} = -\infty$$

$$\lim_{x \to 1^+} \frac{x+1}{x-1} = \frac{2^+}{0^+} = +\infty$$

Since the one-sided limits disagree, $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 - 2x + 1}$ **does not exist**.

Caution: Getting $\dfrac{0}{0}$ after factoring and cancelling means the original limit does not exist (the remaining zero in the denominator is a true vertical asymptote, not a removable one). Getting $\dfrac{c}{0}$ for $c \neq 0$ after simplification is *not* an indeterminate form—it signals infinite behaviour, not a finite limit.

Practice problems. Evaluate each limit, or state that it does not exist.

1. $\lim_{x\to 5} \frac{x^2-25}{x-5}$

2. $\lim_{x\to 3} \frac{x^2-x-6}{x-3}$

3. $\lim_{x\to -4} \frac{x^2+6x+8}{x^2-16}$

4. $\lim_{x\to 2} \frac{x^3-8}{x-2}$

5. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

6. $\lim_{x\to -3} \frac{x^2+6x+9}{x^2+x-6}$

7. $\lim_{x\to 0} \frac{x^3+3x^2}{x^2}$

8. $\lim_{x\to -1} \frac{x^3+x^2+x+1}{x+1}$

9. $\lim_{x\to 2} \frac{x^2-3x+2}{x^2-4x+4}$

10. $\lim_{x\to 3} \frac{x^2-9}{x^2-6x+9}$

Answers:

1. 10 2. 5 3. $\frac{1}{8}$ 4. 12 5. $\frac{3}{2}$ 6. 0 7. 3 8. 2 9. DNE 10. DNE