

# Laplace transforms



## Definition

Let  $f$  be a function defined for  $t \geq 0$ . The **Laplace transform** of  $f$  is:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} f(t) dt$$

provided the integral converges. We write  $\mathcal{L}\{f(t)\} = F(s)$ . The transform converts a function of  $t$  into a function of  $s$ .

**Existence:** If  $f$  is piecewise continuous on  $[0, \infty)$  and of exponential order  $c$  (i.e.  $|f(t)| \leq Me^{ct}$  for large  $t$ ), then  $\mathcal{L}\{f(t)\}$  exists for  $s > c$ .

**Linearity:**  $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

## Table of basic transforms

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$	$\sin kt$	$\frac{k}{s^2 + k^2}$
$t^n$	$\frac{n!}{s^{n+1}}$	$\cos kt$	$\frac{s}{s^2 + k^2}$
$e^{at}$	$\frac{1}{s - a}$	$\sinh kt$	$\frac{k}{s^2 - k^2}$
$t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}}$	$\cosh kt$	$\frac{s}{s^2 - k^2}$
$e^{at} \sin kt$	$\frac{k}{(s - a)^2 + k^2}$	$U(t - a)$	$\frac{e^{-as}}{s}$
$e^{at} \cos kt$	$\frac{s - a}{(s - a)^2 + k^2}$	$f(t - a)U(t - a)$	$e^{-as}F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$

## Transforms of derivatives

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

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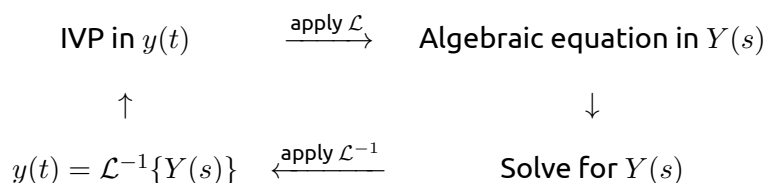
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**Note:** These formulas automatically incorporate the initial conditions, which is why the Laplace method is especially powerful for IVPs.

## Solving an IVP with Laplace transforms



### Strategy:

1. Apply  $\mathcal{L}$  to both sides of the DE. Use the derivative transform formulas to replace  $y'$ ,  $y''$ , etc., substituting the initial conditions immediately.
2. Solve the resulting algebraic equation for  $Y(s)$ .
3. Use partial fractions and the transform table to find  $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ .

**Example:** Solve  $y'' - 3y' + 2y = e^{4t}$ ,  $y(0) = 1$ ,  $y'(0) = 0$ .

### Step 1 — Apply $\mathcal{L}$ :

$$\begin{aligned}
 [s^2Y - s(1) - 0] - 3[sY - 1] + 2Y &= \frac{1}{s - 4} \\
 Y(s^2 - 3s + 2) - s + 3 &= \frac{1}{s - 4}
 \end{aligned}$$

### Step 2 — Solve for $Y$ :

$$Y = \frac{1}{(s - 4)(s - 1)(s - 2)} + \frac{s - 3}{(s - 1)(s - 2)}$$

**Step 3 — Partial fractions and inverse:** Decompose each term over the linear factors, match the table entries, and apply  $\mathcal{L}^{-1}$ .

## First translation theorem (s-shift)

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

$$\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$$

To apply the inverse: complete the square in the denominator to expose the form  $(s - a)^2 + k^2$  or  $(s - a)^2 - k^2$ , then read off the shift.

**Example:** Evaluate  $\mathcal{L}^{-1}\left\{\frac{5s}{s^2 - 2s - 8}\right\}$ .

**Solution:** Factor the denominator:  $s^2 - 2s - 8 = (s - 1)^2 - 9$ . Rewrite  $5s = 5(s - 1) + 5$ :

$$\frac{5s}{(s - 1)^2 - 9} = 5 \cdot \frac{s - 1}{(s - 1)^2 - 9} + \frac{5}{3} \cdot \frac{3}{(s - 1)^2 - 9}$$

$$\mathcal{L}^{-1}\{\dots\} = 5e^t \cosh 3t + \frac{5}{3}e^t \sinh 3t$$

## Unit step function and second translation theorem (t-shift)

The **unit step function** (Heaviside function) switches on at  $t = a$ :

$$U(t - a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq a \end{cases}$$

A piecewise function can be rewritten using  $U$ :

$$f(t) = \begin{cases} g(t) & 0 \leq t < a \\ h(t) & t \geq a \end{cases} = g(t) + [h(t) - g(t)]U(t - a)$$

**Second translation theorem:**

$$\mathcal{L}\{f(t - a)U(t - a)\} = e^{-as}F(s) \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)U(t - a)$$

**Note:** To use the inverse form, the factor  $e^{-as}$  must multiply an expression  $F(s)$  whose inverse  $f(t)$  is known. The result is  $f$  shifted right by  $a$  and switched on at  $t = a$ .

**Example:** Evaluate  $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2 + s - 2}\right\}$ .

**Solution:** Ignore  $e^{-2s}$  temporarily. Factor:  $s^2 + s - 2 = (s - 1)(s + 2)$ . Partial fractions:

$$\frac{1}{(s - 1)(s + 2)} = \frac{1/3}{s - 1} - \frac{1/3}{s + 2} \Rightarrow f(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

Restore the shift ( $a = 2$ ):  $\mathcal{L}^{-1}\{\dots\} = f(t - 2)U(t - 2) = \frac{1}{3}(e^{t-2} - e^{-2(t-2)})U(t - 2)$

## Derivatives of transforms (multiplication by $t$ )

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

**Example:**  $\mathcal{L}\{t \sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2 + 4}\right) = \frac{4s}{(s^2 + 4)^2}$

## Convolution

The **convolution** of  $f$  and  $g$  is:

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

Note that  $f * g = g * f$ . The convolution theorem converts a product of transforms into a convolution in the time domain:

$$\mathcal{L}\{f * g\} = F(s)G(s) \quad \mathcal{L}^{-1}\{F(s)G(s)\} = f * g$$

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**Caution:**  $\mathcal{L}\{f(t) \cdot g(t)\} \neq F(s)G(s)$ . The convolution theorem applies to the *convolution*  $f * g$ , not the pointwise product.

**Special case — Laplace transform of an integral:**

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s} \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau) d\tau$$

**Example:** Evaluate  $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$  using the integral formula.

**Solution:** Write  $\frac{1}{s(s^2+1)} = \frac{1}{s} \cdot \frac{1}{s^2+1}$ . Since  $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ :

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin \tau d\tau = [-\cos \tau]_0^t = 1 - \cos t$$

## Solving IVPs with discontinuous forcing

When the forcing term  $g(t)$  is piecewise, rewrite it using unit step functions before applying  $\mathcal{L}$ .

**Example:** Solve  $y'' + 2y' - 3y = g(t)$  where  $g(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 5 & t \geq 2 \end{cases}$ ,  $y(0) = 0$ ,  $y'(0) = 0$ .

**Step 1 — Rewrite  $g(t)$ :**  $g(t) = 1 + 4U(t-2)$

**Step 2 — Apply  $\mathcal{L}$ :**

$$s^2Y + 2sY - 3Y = \frac{1}{s} + \frac{4e^{-2s}}{s}$$

$$Y = \frac{1}{s(s^2+2s-3)} + \frac{4e^{-2s}}{s(s^2+2s-3)} = \frac{1}{s(s+3)(s-1)} + \frac{4e^{-2s}}{s(s+3)(s-1)}$$

**Step 3 — Partial fractions:** Both terms share the same rational factor  $H(s) = \frac{1}{s(s+3)(s-1)}$ .

Decompose:

$$H(s) = -\frac{1}{3} \cdot \frac{1}{s} + \frac{1}{12} \cdot \frac{1}{s+3} + \frac{1}{4} \cdot \frac{1}{s-1} \implies h(t) = -\frac{1}{3} + \frac{1}{12}e^{-3t} + \frac{1}{4}e^t$$

**Step 4 — Restore the shift:**

$$y(t) = h(t) + 4h(t-2)U(t-2)$$

## Summary of key properties

Property	Formula
Linearity	$\mathcal{L}\{\alpha f + \beta g\} = \alpha F(s) + \beta G(s)$
Derivative	$\mathcal{L}\{f^{(n)}\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$
First translation ( $s$ -shift)	$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$
Second translation ( $t$ -shift)	$\mathcal{L}\{f(t - a)U(t - a)\} = e^{-as} F(s)$
Multiplication by $t^n$	$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
Convolution	$\mathcal{L}\{f * g\} = F(s)G(s)$
Transform of integral	$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s}$