Trigonometric substitution

When to use trigonometric substitution

Use trig substitution when the integrand contains a square root of a quadratic expression that cannot be simplified by ordinary substitution. The three forms and their corresponding substitutions are:

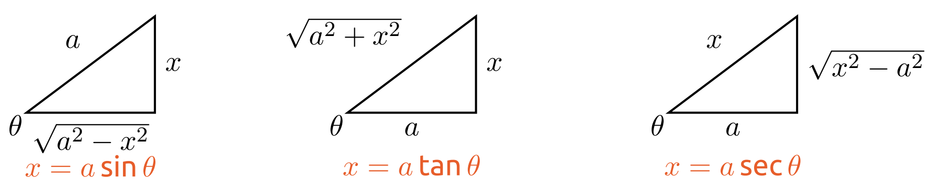
Expression	**Substitution**	**Identity used**	**Restriction on θ**
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$0 \le \theta < \frac{\pi}{2}$ (if $x > 0$)

Strategy:

1. Identify the form and choose the substitution.
2. Substitute for x and dx; simplify the square root using the trig identity.
3. Evaluate the resulting trig integral.
4. **Back-substitute** using a reference triangle to express the answer in terms of x.

Reference triangles

After integrating, back-substitution requires expressing trig functions back in terms of x. Draw a right triangle consistent with the substitution, label two sides, and read off all ratios.



Case 1: $\sqrt{a^2 - x^2}$, substitute $x = a\sin\theta$

Example: Evaluate $\int \frac{1}{x^2\sqrt{9 - x^2}}\, dx$.

Solution: Here $a = 3$, so let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$. Then:

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$$

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$$\int \frac{1}{9\sin^2\theta \cdot 3\cos\theta} \cdot 3\cos\theta \, d\theta = \int \frac{1}{9\sin^2\theta} \, d\theta = \frac{1}{9}\int \csc^2\theta \, d\theta = -\frac{\cot\theta}{9} + C$$

Back-substitute: From the reference triangle with $x = 3\sin\theta$: the opposite side is x, hypotenuse is 3, so the adjacent side is $\sqrt{9-x^2}$. Thus $\cot\theta = \frac{\sqrt{9-x^2}}{x}$.

$$\int \frac{1}{x^2\sqrt{9-x^2}} \, dx = -\frac{\sqrt{9-x^2}}{9x} + C$$

Case 2: $\sqrt{a^2+x^2}$, substitute $x = a\tan\theta$

Example: Evaluate $\int \frac{\sqrt{4+x^2}}{x} \, dx$.

Solution: Let $x = 2\tan\theta$, $dx = 2\sec^2\theta \, d\theta$. Then:

$$\sqrt{4+x^2} = \sqrt{4\sec^2\theta} = 2\sec\theta$$

$$\int \frac{2\sec\theta}{2\tan\theta} \cdot 2\sec^2\theta \, d\theta = 2\int \frac{\sec^3\theta}{\tan\theta} \, d\theta = 2\int \frac{\sec^2\theta \cdot \sec\theta}{\tan\theta} \, d\theta$$

Write $\sec^2\theta = 1 + \tan^2\theta$ and split:

$$2\int \frac{(1+\tan^2\theta)\sec\theta}{\tan\theta} \, d\theta = 2\int \frac{\sec\theta}{\tan\theta} \, d\theta + 2\int \sec\theta\tan\theta \, d\theta = 2\int \csc\theta \, d\theta + 2\sec\theta$$

$$= 2\ln|\csc\theta - \cot\theta| + 2\sec\theta + C$$

Back-substitute: From the triangle, $\sec\theta = \frac{\sqrt{4+x^2}}{2}$, $\csc\theta = \frac{\sqrt{4+x^2}}{x}$, $\cot\theta = \frac{2}{x}$:

$$\int \frac{\sqrt{4+x^2}}{x} \, dx = \sqrt{4+x^2} + 2\ln\left|\frac{\sqrt{4+x^2}-2}{x}\right| + C$$

Case 3: $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$

Example: Evaluate $\int \frac{1}{\sqrt{x^2-16}} \, dx$.

Solution: Let $x = 4\sec\theta$, $dx = 4\sec\theta\tan\theta \, d\theta$. Then:

$$\sqrt{x^2-16} = \sqrt{16\tan^2\theta} = 4\tan\theta$$

$$\int \frac{4\sec\theta\tan\theta}{4\tan\theta} \, d\theta = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Back-substitute: From the triangle, $\sec\theta = \frac{x}{4}$ and $\tan\theta = \frac{\sqrt{x^2-16}}{4}$:

$$\int \frac{1}{\sqrt{x^2-16}} \, dx = \ln\left|\frac{x+\sqrt{x^2-16}}{4}\right| + C$$