Integration by substitution

The idea

Substitution is the integration counterpart of the **chain rule**. If the integrand contains a composite function $f(g(x))$ multiplied by $g'(x)$, the substitution $u = g(x)$ transforms the integral into a simpler one in terms of u:

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

The key step is replacing *every* x in the integrand—including dx—with expressions in u.

Strategy

1. **Choose** u: look for an "inside" function whose derivative also appears (up to a constant factor).
2. **Differentiate:** compute $\frac{du}{dx} = g'(x)$, then write $du = g'(x)\, dx$ and solve for dx.
3. **Substitute:** replace all x-expressions with u-expressions.
4. **Integrate** in u.
5. **Back-substitute:** replace u with $g(x)$ to express the answer in terms of x.

Note: After substituting, no x should remain in the integrand. If x is still present, either the substitution is incomplete or a different u is needed.

Basic examples

Example: Evaluate $\int (3x+1)^5\, dx$.

Solution: The inside function is $3x+1$. Let $u = 3x+1$, so $du = 3\, dx$, giving $dx = \frac{du}{3}$:

$$\int u^5 \cdot \frac{du}{3} = \frac{1}{3} \cdot \frac{u^6}{6} + C = \frac{(3x+1)^6}{18} + C$$

Example: Evaluate $\int x\sqrt{x^2+1}\, dx$.

Solution: The inside function is x^2+1, whose derivative $2x$ is present up to a factor. Let $u = x^2+1$, $du = 2x\, dx$, so $x\, dx = \frac{du}{2}$:

$$\int \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C$$

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Example: Evaluate $\int \frac{e^x}{1+e^x}\,dx$.
Solution: Let $u = 1 + e^x$, $du = e^x\,dx$:

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln(1+e^x) + C$$

When the match is off by a constant

The derivative of u often appears in the integrand multiplied by the wrong constant. Factor it out to fix the match.
Example: Evaluate $\int x^2 \cos(x^3)\,dx$.

Solution: Let $u = x^3$, $du = 3x^2\,dx$, so $x^2\,dx = \frac{du}{3}$:

$$\int \cos(u)\cdot\frac{du}{3} = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3) + C$$

Note: You can only adjust for a *constant* mismatch. If the leftover factor contains x, substitution alone will not work—a different technique is needed.

Definite integrals: changing the limits

For definite integrals, convert the limits of integration to u-values so you do **not** need to back-substitute.

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example: Evaluate $\int_0^2 x\,e^{x^2}\,dx$.

Solution: Let $u = x^2$, $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$.
New limits: when $x = 0$, $u = 0$; when $x = 2$, $u = 4$.

$$\int_0^4 e^u\cdot\frac{du}{2} = \frac{1}{2}\Big[e^u\Big]_0^4 = \frac{1}{2}(e^4 - 1)$$

Example: Evaluate $\int_1^2 \frac{\ln x}{x}\,dx$.

Solution: Let $u = \ln x$, $du = \frac{1}{x}\,dx$.
New limits: when $x = 1$, $u = 0$; when $x = 2$, $u = \ln 2$.

$$\int_0^{\ln 2} u\,du = \left[\frac{u^2}{2}\right]_0^{\ln 2} = \frac{(\ln 2)^2}{2}$$

Integrals requiring algebraic preparation

Sometimes the integrand must be rewritten before the correct u becomes visible.

Example: Evaluate $\int \frac{x+3}{x^2+6x+1}\,dx$.

Solution: Observe that $(x^2+6x+1)' = 2x+6 = 2(x+3)$. The numerator is exactly half the derivative of the denominator. Let $u = x^2+6x+1$, $du = (2x+6)\,dx = 2(x+3)\,dx$, so $(x+3)\,dx = \frac{du}{2}$:

$$\int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+6x+1| + C$$

Example: Evaluate $\int \frac{1}{x\ln x}\,dx$.

Solution: The factor $\frac{1}{x}$ is the derivative of $\ln x$. Let $u = \ln x$, $du = \frac{dx}{x}$:

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C$$

Common substitutions to recognize

Integrand contains	Try	Reason
$(ax+b)^n$	$u = ax+b$	Power of linear function
$e^{g(x)}\cdot g'(x)$	$u = g(x)$	Exponential of inside
$\frac{g'(x)}{g(x)}$	$u = g(x)$	Gives $\int \frac{1}{u}\,du = \ln|u|$
$g'(x)\cdot[g(x)]^n$	$u = g(x)$	Power of inside function
$\sin(g(x))\cdot g'(x)$	$u = g(x)$	Trig of inside function
$\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$	See: trig substitution

Caution: Not every integral yields to substitution. If no choice of u eliminates all x-terms, consider integration by parts, partial fractions, or trig substitution instead.