

Integration by parts



The formula

Integration by parts is derived from the **product rule** for differentiation. Rearranging and integrating $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$ gives:

$$\int u dv = uv - \int v du$$

Use this when the integrand is a **product of two functions** and substitution does not apply.

Choosing u and dv : the LIATE guide

Choose u as the function appearing **earlier** in this list; the rest becomes dv :

- L** Logarithmic: $\ln x, \log_a x$
- I** Inverse trig: $\arctan x, \arcsin x, \dots$
- A** Algebraic (polynomials): x^n
- T** Trigonometric: $\sin x, \cos x, \dots$
- E** Exponential: e^x, a^x

Note: LIATE is a *guide*, not a rule. The circular case below is a notable exception.

Basic examples

Example: Evaluate $\int x \cos x dx$.

Solution: Polynomial (**A**) before trig (**T**), so $u = x$:

$$u = x, \quad dv = \cos x dx, \quad du = dx, \quad v = \sin x$$

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C$$

Example: Evaluate $\int \ln x dx$.

Solution: When the integrand is a **single L** or **I** function, set $dv = dx$ (choose u as the whole integrand):

$$u = \ln x, \quad dv = dx, \quad du = \frac{1}{x} dx, \quad v = x$$

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - x + C$$

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Repeated integration by parts

When $u = x^n$ with $n \geq 2$, apply by-parts n times, reducing the power each time.

Example: Evaluate $\int x^2 e^x dx$.

Solution: First application: $u = x^2$, $dv = e^x dx$, so $du = 2x dx$, $v = e^x$:

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

Second application on $\int x e^x dx$: $u = x$, $dv = e^x dx$, so $du = dx$, $v = e^x$:

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x$$

Substituting back:

$$\int x^2 e^x dx = x^2 e^x - 2(x e^x - e^x) + C = e^x(x^2 - 2x + 2) + C$$

Note: Add $+C$ only at the very end, not after each intermediate step.

The circular case

Integrals of the form $\int e^x \sin x dx$ and $\int e^x \cos x dx$ reproduce themselves after two applications of by-parts. This allows you to **solve for the integral algebraically**.

Example: Evaluate $\int e^x \sin x dx$.

Solution: Let $I = \int e^x \sin x dx$.

First application: $u = \sin x$, $dv = e^x dx$, so $du = \cos x dx$, $v = e^x$:

$$I = e^x \sin x - \int e^x \cos x dx$$

Second application on $\int e^x \cos x dx$: $u = \cos x$, $dv = e^x dx$, so $du = -\sin x dx$, $v = e^x$:

$$I = e^x \sin x - \left(e^x \cos x + \int e^x \sin x dx \right) = e^x \sin x - e^x \cos x - I$$

Adding I to both sides and solving:

$$2I = e^x(\sin x - \cos x) \Rightarrow I = \frac{e^x(\sin x - \cos x)}{2} + C$$

Caution: Make the **same assignment** for u at both steps. Alternating choices (e.g., $u = \sin x$ then $u = e^x$) undoes the first step and yields $0 = 0$.

Definite integrals

Apply limits to the boundary term and keep the remaining integral definite:

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

Example: Evaluate $\int_1^e x \ln x \, dx$.

Solution: Logarithm (**L**) before algebraic (**A**), so $u = \ln x$:

$$u = \ln x, \quad dv = x \, dx, \quad du = \frac{1}{x} \, dx, \quad v = \frac{x^2}{2}$$

$$\begin{aligned} \int_1^e x \ln x \, dx &= \left[\frac{x^2}{2} \ln x \right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} \, dx \\ &= \left(\frac{e^2}{2} \cdot 1 - \frac{1}{2} \cdot 0 \right) - \frac{1}{2} \int_1^e x \, dx \\ &= \frac{e^2}{2} - \frac{1}{2} \left[\frac{x^2}{2} \right]_1^e \\ &= \frac{e^2}{2} - \frac{1}{2} \cdot \frac{e^2 - 1}{2} = \frac{e^2}{2} - \frac{e^2 - 1}{4} = \frac{e^2 + 1}{4} \end{aligned}$$

Summary of strategies

Integrand type	Strategy
$x^n e^x, x^n \sin x, x^n \cos x$	$u = x^n$; apply n times
$\ln x, \arctan x, \arcsin x$ alone	$u =$ the function, $dv = dx$
$x^n \ln x, x^n \arctan x$	$u = \ln x$ or $u = \arctan x$
$e^x \sin x, e^x \cos x$	Apply twice; solve for I algebraically

Caution: If applying by-parts makes the integral *more* complicated, reconsider your choice of u and dv .