Partial fraction decomposition

When to use partial fractions

Use partial fraction decomposition to integrate a **rational function** $\frac{P(x)}{Q(x)}$, where P and Q are polynomials.

Before you start:

1. **Check degree:** If deg$(P) \geq$ deg(Q), perform **polynomial long division** first, then decompose the remainder.
2. **Factor the denominator** $Q(x)$ completely over the reals.
3. **Set up** the partial fraction form based on the factor types below.

Note: Partial fractions only apply when the integrand is a *proper* rational function (degree of numerator strictly less than degree of denominator). Always check this first.

Setting up the decomposition

The form of the partial fraction decomposition depends on the **type of factors** in $Q(x)$:

Factor in $Q(x)$	**Multiplicity**	**Partial fraction terms**
Linear: $(x-a)$	once	$\frac{A}{x-a}$
Linear: $(x-a)^2$	repeated	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
Linear: $(x-a)^k$	k times	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$
Irreducible quadratic: (x^2+bx+c)	once	$\frac{Ax+B}{x^2+bx+c}$
Irreducible quadratic: $(x^2+bx+c)^k$	k times	$\frac{A_1x+B_1}{x^2+bx+c} + \cdots + \frac{A_kx+B_k}{(x^2+bx+c)^k}$

Finding the constants

After setting up the decomposition, multiply both sides by $Q(x)$ to clear denominators, then use one or both methods:

Method 1 — Substitution (cover-up method): Plug in values of x that make a factor zero. This quickly gives constants for linear factors.

Method 2 — Expand and match coefficients: Expand the right side, collect by powers of x, and equate coefficients of matching powers. This is required when irreducible quadratics are present.

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In practice, use **both methods together**: substitution to find as many constants as possible, then match coefficients for the rest.

Integrating each term

After decomposing, each term has a standard integral:

Term	Integral
$\dfrac{A}{x-a}$	$A \ln\|x-a\| + C$
$\dfrac{A}{(x-a)^k}$, $k \geq 2$	$\dfrac{A}{(1-k)(x-a)^{k-1}} + C$
$\dfrac{A}{x^2+a^2}$	$\dfrac{A}{a}\arctan\left(\dfrac{x}{a}\right) + C$
$\dfrac{Ax+B}{x^2+bx+c}$	Complete the square; split into ln (via substitution) and arctan terms

Case 1: Distinct linear factors

Example: Evaluate $\displaystyle\int \frac{3x+5}{(x-1)(x+3)}\,dx$.
Solution: Both factors are distinct and linear, so write:

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

Multiply both sides by $(x-1)(x+3)$:

$$3x+5 = A(x+3) + B(x-1)$$

Substitution:

- $x=1$: $8 = 4A \implies A = 2$
- $x=-3$: $-4 = -4B \implies B = 1$

Therefore:

$$\int \frac{3x+5}{(x-1)(x+3)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{1}{x+3}\,dx$$
$$= 2\ln|x-1| + \ln|x+3| + C$$

Case 2: Repeated linear factor

Example: Evaluate $\displaystyle\int \frac{x^2+4}{x(x-2)^2}\,dx$.

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Solution: The factor $(x-2)$ appears twice, so write:

$$\frac{x^2+4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Multiply both sides by $x(x-2)^2$:

$$x^2+4 = A(x-2)^2 + Bx(x-2) + Cx$$

Substitution:

- $x=0$: $4 = 4A \implies A = 1$
- $x=2$: $8 = 2C \implies C = 4$

Match coefficients (to find B). Expanding the right side:

$$A(x^2-4x+4) + B(x^2-2x) + Cx = (A+B)x^2 + (-4A-2B+C)x + 4A$$

Matching x^2: $1 = A+B \implies B = 0$.
Therefore:

$$\begin{aligned}\int \frac{x^2+4}{x(x-2)^2}\,dx &= \int \frac{1}{x}\,dx + \int \frac{4}{(x-2)^2}\,dx \\ &= \ln|x| - \frac{4}{x-2} + C\end{aligned}$$

Case 3: Irreducible quadratic factor

Example: Evaluate $\int \frac{5x^2+7x+6}{(x+1)(x^2+1)}\,dx$.
Solution: The quadratic x^2+1 is irreducible (discriminant $= -4 < 0$), so write:

$$\frac{5x^2+7x+6}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $(x+1)(x^2+1)$:

$$5x^2+7x+6 = A(x^2+1) + (Bx+C)(x+1)$$

Substitution:

- $x=-1$: $4 = 2A \implies A = 2$

Match coefficients (expand the right side):

$$Ax^2 + A + Bx^2 + Bx + Cx + C = (A+B)x^2 + (B+C)x + (A+C)$$

- x^2: $5 = A+B \implies B = 3$

- constant: $6 = A + C \implies C = 4$

Therefore:

$$\int \frac{5x^2+7x+6}{(x+1)(x^2+1)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3x+4}{x^2+1}\,dx$$
$$= 2\ln|x+1| + \int \frac{3x}{x^2+1}\,dx + \int \frac{4}{x^2+1}\,dx$$

For the middle term, use substitution $u = x^2 + 1$, $du = 2x\,dx$:

$$\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln\left(x^2+1\right)$$

Therefore:

$$\int \frac{5x^2+7x+6}{(x+1)(x^2+1)}\,dx = 2\ln|x+1| + \frac{3}{2}\ln\left(x^2+1\right) + 4\arctan(x) + C$$

Improper rational functions: long division first

If $\deg(P) \geq \deg(Q)$, divide first, then decompose the remainder.

Example: Evaluate $\int \frac{x^3-x+2}{x^2-x-2}\,dx$.

Solution: Since the degree of the numerator is greater than or equal to the degree of the denominator, divide $x^3 - x + 2$ by $x^2 - x - 2$:

$$x^3 - x + 2 = (x^2 - x - 2)(x+1) + (2x+4)$$

So:

$$\frac{x^3-x+2}{x^2-x-2} = x + 1 + \frac{2x+4}{x^2-x-2}$$

Now factor the denominator: $x^2 - x - 2 = (x-2)(x+1)$, and decompose:

$$\frac{2x+4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

Multiplying through: $2x + 4 = A(x+1) + B(x-2)$.

- $x = 2$: $8 = 3A \implies A = \frac{8}{3}$
- $x = -1$: $2 = -3B \implies B = -\frac{2}{3}$

Therefore:

$$\int \frac{x^3-x+2}{x^2-x-2}\,dx = \int \left(x + 1 + \frac{8/3}{x-2} - \frac{2/3}{x+1}\right)dx$$
$$= \frac{x^2}{2} + x + \frac{8}{3}\ln|x-2| - \frac{2}{3}\ln|x+1| + C$$