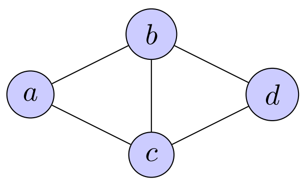
Graphs

Definition

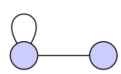
A **graph** $G = (V, E)$ consists of a finite set of **vertices** V and a set of **edges** E connecting pairs of vertices.
The **degree** of a vertex, denoted deg(v), is the number of edges incident to it. If two vertices are connected by an edge, they are **adjacent** and are called **neighbours**.

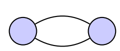


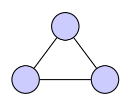
$V = \{a, b, c, d\}$
$E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}\}$
deg$(a) = 2$, deg$(b) = 3$, deg$(c) = 3$, deg$(d) = 2$

Simple graphs

Term	Definition
Loop	An edge from a vertex to itself
Multiple edge	Two or more edges connecting the same pair of vertices
Simple graph	A graph with no loops and no multiple edges


Has a loop
(not simple)


Multiple edges
(not simple)


Simple graph

Handshaking Theorem

The sum of all vertex degrees equals twice the number of edges:

$$\sum_{v \in V} \text{deg}(v) = 2|E|$$

Corollary: The number of vertices with odd degree is always even.
Example: A graph has 5 vertices with degrees 3, 3, 2, 2, 2. How many edges?
$\sum \text{deg}(v) = 3 + 3 + 2 + 2 + 2 = 12$, so $|E| = 12/2 = 6$ edges.

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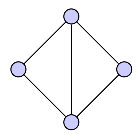
Degree sequence

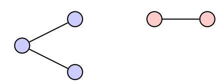
The **degree sequence** of a graph is the list of vertex degrees in non-increasing order.
Example: The graph above has degree sequence $(3, 3, 2, 2, 2)$.
Note: In a simple graph with n vertices, the maximum degree is $n - 1$ (a vertex can connect to all others but not itself).

Connected graphs

A graph is **connected** if there is a path between every pair of vertices. Otherwise, it is **disconnected**.
A **component** is a maximal connected subgraph of a disconnected graph.

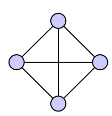


Connected

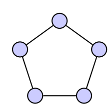


Disconnected (2 components)

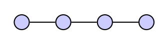
Common graph families



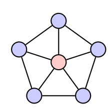
K_4: Complete
(every pair connected)



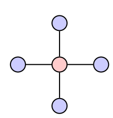
C_5: Cycle
(single closed loop)



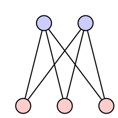
P_4: Path
(no cycles)



W_5: Wheel
(cycle + center)



$K_{1,4}$: Star
(one center vertex)



$K_{2,3}$: Complete bipartite
(two groups, all cross-edges)

Summary of common graphs

Graph	Vertices	Edges	Degrees
Complete K_n	n	$\frac{n(n-1)}{2}$	All vertices have degree $n-1$
Cycle C_n	n	n	All vertices have degree 2
Path P_n	n	$n-1$	Two endpoints degree 1, rest degree 2
Wheel W_n	$n+1$	$2n$	Center degree n, rim degree 3
Star $K_{1,n}$	$n+1$	n	Center degree n, leaves degree 1
Complete bipartite $K_{m,n}$	$m+n$	mn	Degrees n and m