Fourier transforms

From Fourier series to Fourier transform

Fourier series represent **periodic** functions. For **non-periodic** functions defined on $(-\infty, \infty)$, we use the Fourier transform.
Think of it as taking $L \to \infty$: the discrete frequencies $\frac{n\pi}{L}$ become a continuous spectrum.

Definition

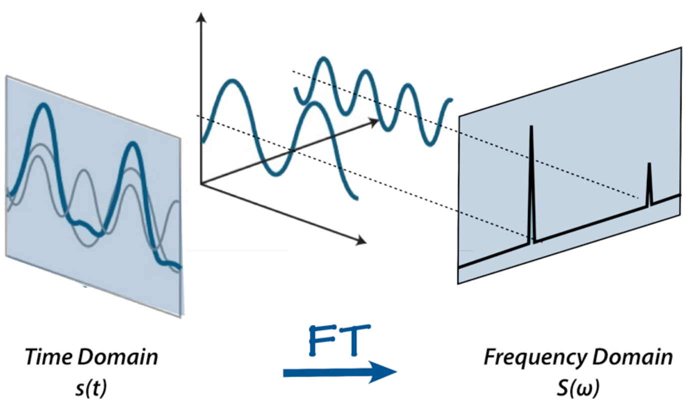
The **Fourier transform** of $f(x)$ is:

$$\mathcal{F}\{f(x)\} = F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

The **inverse Fourier transform** recovers $f(x)$:

$$\mathcal{F}^{-1}\{F(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

Note: Different textbooks use different conventions for where to place 2π and the sign in the exponent. Always check which convention your course uses.



Alternative symmetric convention:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

Existence conditions

The Fourier transform exists if:

- $f(x)$ is absolutely integrable: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$
- $f(x)$ is piecewise continuous

Common Fourier transform pairs

$f(x)$	$F(\omega) = \mathcal{F}\{f(x)\}$
$e^{-a\|x\|} \quad (a > 0)$	$\dfrac{2a}{a^2+\omega^2}$
$e^{-ax^2} \quad (a > 0)$	$\sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/(4a)}$
$\text{rect}(x) = \begin{cases} 1 & \|x\| < \frac{1}{2} \\ 0 & \|x\| > \frac{1}{2} \end{cases}$	$\text{sinc}\left(\dfrac{\omega}{2}\right) = \dfrac{\sin(\omega/2)}{\omega/2}$
$\delta(x)$	1
1	$2\pi\delta(\omega)$
$e^{i\omega_0 x}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 x)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 x)$	$\dfrac{\pi}{i}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

Properties of Fourier transforms

Let $\mathcal{F}\{f(x)\} = F(\omega)$ and $\mathcal{F}\{g(x)\} = G(\omega)$.

Property	**Time domain**	**Frequency domain**
Linearity	$af(x)+bg(x)$	$aF(\omega)+bG(\omega)$
Time shift	$f(x-x_0)$	$e^{-i\omega x_0}F(\omega)$
Frequency shift	$e^{i\omega_0 x}f(x)$	$F(\omega-\omega_0)$
Scaling	$f(ax)$	$\dfrac{1}{\|a\|}F\left(\dfrac{\omega}{a}\right)$
Time derivative	$f'(x)$	$i\omega F(\omega)$
n-th derivative	$f^{(n)}(x)$	$(i\omega)^n F(\omega)$
Multiplication by x	$xf(x)$	$i\dfrac{dF}{d\omega}$
Convolution	$(f*g)(x)$	$F(\omega)\cdot G(\omega)$
Multiplication	$f(x)\cdot g(x)$	$\dfrac{1}{2\pi}(F*G)(\omega)$

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Example: Computing a Fourier transform

Find $\mathcal{F}\{e^{-3|x|}\}$.
Solution: Using the formula with $a = 3$:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-3|x|} e^{-i\omega x}\,dx = \int_{-\infty}^{0} e^{3x} e^{-i\omega x}\,dx + \int_{0}^{\infty} e^{-3x} e^{-i\omega x}\,dx$$

$$= \int_{-\infty}^{0} e^{(3-i\omega)x}\,dx + \int_{0}^{\infty} e^{-(3+i\omega)x}\,dx$$

$$= \frac{e^{(3-i\omega)x}}{3-i\omega}\Big|_{-\infty}^{0} + \frac{e^{-(3+i\omega)x}}{-(3+i\omega)}\Big|_{0}^{\infty}$$

$$= \frac{1}{3-i\omega} + \frac{1}{3+i\omega} = \frac{(3+i\omega)+(3-i\omega)}{(3-i\omega)(3+i\omega)} = \frac{6}{9+\omega^2}$$

This matches the table entry $\dfrac{2a}{a^2+\omega^2}$ with $a = 3$.

Example: Using properties

Find $\mathcal{F}\{xe^{-2|x|}\}$.
Solution: Use the multiplication by x property: $\mathcal{F}\{xf(x)\} = i\dfrac{dF}{d\omega}$.
We know $\mathcal{F}\{e^{-2|x|}\} = \dfrac{4}{4+\omega^2}$.

$$\mathcal{F}\{xe^{-2|x|}\} = i\frac{d}{d\omega}\left(\frac{4}{4+\omega^2}\right) = i \cdot \frac{-8\omega}{(4+\omega^2)^2} = \frac{-8i\omega}{(4+\omega^2)^2}$$

Convolution

The **convolution** of f and g is:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

Convolution theorem: Convolution in the time domain becomes multiplication in the frequency domain:

$$\mathcal{F}\{f * g\} = F(\omega) \cdot G(\omega)$$

This is extremely useful for solving differential equations and analyzing linear systems.

Parseval's theorem

Energy is preserved between domains:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Solving ODEs with Fourier transforms

Fourier transforms convert derivatives to algebraic multiplication, making differential equations easier to solve.

Example: Solve $y'' - 4y = e^{-|x|}$.

Step 1: Take the Fourier transform of both sides.

Using $\mathcal{F}\{y''\} = (i\omega)^2 Y(\omega) = -\omega^2 Y(\omega)$ and $\mathcal{F}\{e^{-|x|}\} = \dfrac{2}{1+\omega^2}$:

$$-\omega^2 Y(\omega) - 4Y(\omega) = \frac{2}{1+\omega^2}$$

Step 2: Solve for $Y(\omega)$.

$$Y(\omega)(-\omega^2 - 4) = \frac{2}{1+\omega^2}$$

$$Y(\omega) = \frac{2}{(1+\omega^2)(-\omega^2-4)} = \frac{-2}{(1+\omega^2)(4+\omega^2)}$$

Step 3: Use partial fractions.

$$\frac{-2}{(1+\omega^2)(4+\omega^2)} = \frac{A}{1+\omega^2} + \frac{B}{4+\omega^2}$$

Solving: $-2 = A(4+\omega^2) + B(1+\omega^2)$

Setting $\omega^2 = -1$: $-2 = 3A \Rightarrow A = -\frac{2}{3}$

Setting $\omega^2 = -4$: $-2 = -3B \Rightarrow B = \frac{2}{3}$

$$Y(\omega) = \frac{-2/3}{1+\omega^2} + \frac{2/3}{4+\omega^2}$$

Step 4: Take the inverse Fourier transform.

Using $\mathcal{F}^{-1}\left\{\dfrac{2a}{a^2+\omega^2}\right\} = e^{-a|x|}$:

$$y(x) = -\frac{2}{3} \cdot \frac{1}{2} e^{-|x|} + \frac{2}{3} \cdot \frac{1}{4} e^{-2|x|} = -\frac{1}{3} e^{-|x|} + \frac{1}{6} e^{-2|x|}$$

Fourier sine and cosine transforms

For functions defined on $[0, \infty)$:

Fourier cosine transform (for even extensions):

$$\mathcal{F}_c\{f(x)\} = F_c(\omega) = \int_0^\infty f(x)\cos(\omega x)\,dx$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$$

Fourier sine transform (for odd extensions):

$$\mathcal{F}_s\{f(x)\} = F_s(\omega) = \int_0^\infty f(x)\sin(\omega x)\,dx$$

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega$$

Derivative properties:

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 F_c(\omega) - f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 F_s(\omega) + \omega f(0)$$

Use the cosine transform when boundary conditions involve $f'(0)$; use the sine transform when they involve $f(0)$.

Summary

Tool	Use for
Fourier series	Periodic functions on $[-L, L]$
Fourier transform	Non-periodic functions on $(-\infty, \infty)$
Fourier cosine transform	Functions on $[0, \infty)$ with even extension
Fourier sine transform	Functions on $[0, \infty)$ with odd extension

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