Fourier series

Overview

A Fourier series represents a periodic function as an infinite sum of sines and cosines. It is essential for wave analysis, heat equations, and solving differential equations.
For $f(x)$ on $[-L, L]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

Fourier coefficients

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Useful values: $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for any integer n.

Orthogonality relations

These identities (on $[-L, L]$) allow us to isolate coefficients:

$$\begin{array}{l|l} \displaystyle\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 & \text{if } m \neq n \\ \displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0 & \text{if } m \neq n \\ \displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0 & \text{for all } m, n \end{array}$$

Even and odd functions

Symmetry simplifies calculations:

Type	Property	Series form	Simplification
Even	$f(-x) = f(x)$	$a_0 + \sum a_n \cos\left(\frac{n\pi x}{L}\right)$	$b_n = 0$
Odd	$f(-x) = -f(x)$	$\sum b_n \sin\left(\frac{n\pi x}{L}\right)$	$a_0 = 0$, $a_n = 0$

Recall: even $\times$ even $=$ even, odd $\times$ odd $=$ even, even $\times$ odd $=$ odd
For even functions: $\int_{-L}^{L} f(x)\, dx = 2\int_{0}^{L} f(x)\, dx$ For odd functions: $\int_{-L}^{L} f(x)\, dx = 0$

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Example: Piecewise function

Find the Fourier series for $f(x) = \begin{cases} 1 & -1 \le x < 0 \\ x & 0 \le x < 1 \end{cases}$ on $[-1, 1]$.

Here $L = 1$.

Find a_0:

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0} 1\,dx + \int_{0}^{1} x\,dx\right] = \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4}$$

Find a_n:

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} x\cos(n\pi x)\,dx$$

First integral: $\int_{-1}^{0} \cos(n\pi x)\,dx = \frac{\sin(n\pi x)}{n\pi}\Big|_{-1}^{0} = 0$

Second integral (by parts): $\int_{0}^{1} x\cos(n\pi x)\,dx = \frac{\cos(n\pi) - 1}{n^2\pi^2} = \frac{(-1)^n - 1}{n^2\pi^2}$

So $a_n = \frac{(-1)^n - 1}{n^2\pi^2}$

Find b_n:

$$b_n = \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} x\sin(n\pi x)\,dx = \frac{-1 + \cos(n\pi)}{n\pi} + \frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2\pi^2}$$

$$= \frac{(-1)^n - 1}{n\pi} + \frac{-n\pi(-1)^n}{n^2\pi^2} = \frac{(-1)^n - 1}{n\pi} - \frac{(-1)^n}{n\pi} = \frac{-1}{n\pi}$$

Fourier series:

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)\right]$$

Example: Even function

Find the Fourier series for $f(x) = x^2$ on $[-2, 2]$.

Since $f(x) = x^2$ is even, $b_n = 0$ for all n. Here $L = 2$.

Find a_0:

$$a_0 = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{1}{4}\cdot 2\int_{0}^{2} x^2\,dx = \frac{1}{2}\cdot\frac{x^3}{3}\Big|_{0}^{2} = \frac{1}{2}\cdot\frac{8}{3} = \frac{4}{3}$$

Find a_n: Using integration by parts twice:

$$a_n = \frac{1}{2}\cdot 2\int_{0}^{2} x^2\cos\left(\frac{n\pi x}{2}\right)dx = \frac{16(-1)^n}{n^2\pi^2}$$

Fourier series:

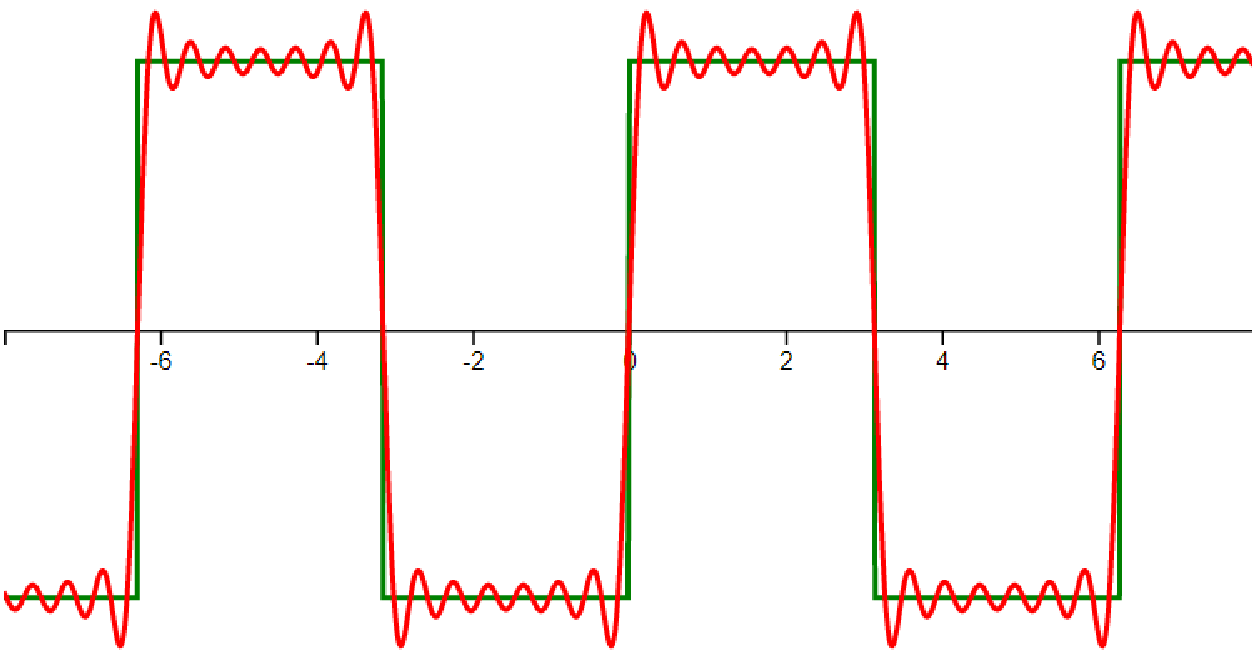
$$f(x) = x^2 = \frac{4}{3} + \sum_{n=1}^{\infty}\frac{16(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$$

Convergence

Fourier Convergence Theorem: If $f(x)$ is piecewise smooth on $[-L, L]$, then the Fourier series converges to:

- $f(x)$ where f is continuous
- $\dfrac{f(x^+) + f(x^-)}{2}$ at jump discontinuities (the midpoint)

Note: Near discontinuities, partial sums exhibit overshoot called **Gibbs phenomenon**.



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