# Domain and range

## Definitions

The **domain** of a function is the set of all possible input values ($x$-values).
The **range** of a function is the set of all possible output values ($y$-values).

## Finding the domain

Look for values of $x$ that would cause problems:

| **Type of function** | **Restriction** | **Example** |
|---|---|---|
| Polynomial | None (all real numbers) | $f(x) = x^3 - 2x + 1$ |
| Rational $\frac{p(x)}{q(x)}$ | Denominator $\neq 0$ | $f(x) = \frac{1}{x-3}$: $x \neq 3$ |
| Square root $\sqrt{g(x)}$ | Radicand $\geq 0$ | $f(x) = \sqrt{x-2}$: $x \geq 2$ |
| Logarithm ln$(g(x))$ | Argument $> 0$ | $f(x) =$ ln$(x+1)$: $x > -1$ |

Example: Find the domain of $f(x) = \frac{1}{\sqrt{x-4}}$.
Two restrictions apply:

- Square root: $x - 4 \geq 0 \Rightarrow x \geq 4$
- Denominator: $\sqrt{x-4} \neq 0 \Rightarrow x \neq 4$

Combining: $x > 4$, so the domain is $(4, \infty)$.

Example: Find the domain of $f(x) = \frac{x+2}{x^2-9}$.
Set denominator $\neq 0$: $x^2 - 9 \neq 0 \Rightarrow (x-3)(x+3) \neq 0 \Rightarrow x \neq 3$ and $x \neq -3$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ or equivalently $\{x \in \mathbb{R} : x \neq \pm 3\}$

## Finding the range

Common strategies:

1. **Graph the function** and identify the $y$-values covered
2. **Solve for** $x$ in terms of $y$ and find restrictions on $y$
3. **Use known transformations** of parent functions

Example: Find the range of $f(x) = x^2 + 3$.
The parent function $y = x^2$ has range $[0, \infty)$. Adding 3 shifts up by 3.
Range: $[3, \infty)$
Example: Find the range of $f(x) = \frac{1}{x-2} + 5$.
The parent function $y = \frac{1}{x}$ has range $(-\infty, 0) \cup (0, \infty)$. Adding 5 shifts up.
Range: $(-\infty, 5) \cup (5, \infty)$ or $\{y \in \mathbb{R} : y \neq 5\}$

## Common functions: Domain and range

| Function | Domain | Range |
|---|---|---|
| $f(x) = c$ (constant) | $(-\infty, \infty)$ | $\{c\}$ |
| $f(x) = x$ (linear) | $(-\infty, \infty)$ | $(-\infty, \infty)$ |
| $f(x) = x^2$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $f(x) = x^3$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ |
| $f(x) = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $f(x) = \sqrt[3]{x}$ | $(-\infty, \infty)$ | $(-\infty, \infty)$ |
| $f(x) = \lvert x \rvert$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $f(x) = \frac{1}{x}$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $f(x) = e^x$ | $(-\infty, \infty)$ | $(0, \infty)$ |
| $f(x) = \ln(x)$ | $(0, \infty)$ | $(-\infty, \infty)$ |
| $f(x) = \sin(x)$ | $(-\infty, \infty)$ | $[-1, 1]$ |
| $f(x) = \cos(x)$ | $(-\infty, \infty)$ | $[-1, 1]$ |
| $f(x) = \tan(x)$ | $\left\{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\right\}$ | $(-\infty, \infty)$ |
| $f(x) = \arcsin(x)$ | $[-1, 1]$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ |
| $f(x) = \arccos(x)$ | $[-1, 1]$ | $[0, \pi]$ |
| $f(x) = \arctan(x)$ | $(-\infty, \infty)$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

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## Trigonometric functions

sin$(x)$ and cos$(x)$ accept any real number as input and oscillate between $-1$ and $1$.
tan$(x) = \dfrac{\sin x}{\cos x}$ is undefined wherever $\cos x = 0$, which occurs at $x = \dfrac{\pi}{2} + n\pi$ for any integer $n$. These are the vertical asymptotes of tan$(x)$.

## Inverse trigonometric functions

Since sin, cos, and tan are not one-to-one over all real numbers, their inverses are only defined on a **restricted domain**. The range of each inverse function reflects the restricted interval on which the original was inverted.

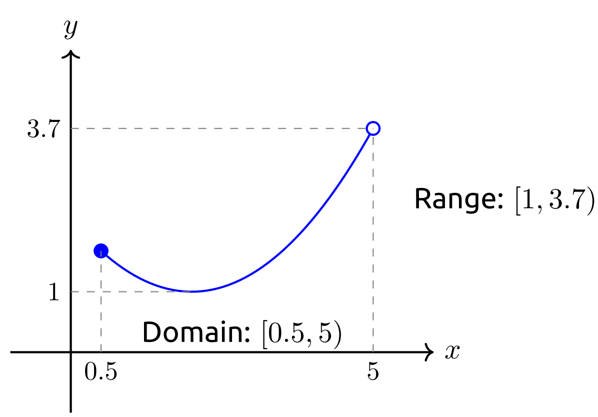
| Function | Read as | Domain | Range |
|---|---|---|---|
| arcsin$(x)$ | "the angle whose sine is $x$" | $[-1, 1]$ | $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ |
| arccos$(x)$ | "the angle whose cosine is $x$" | $[-1, 1]$ | $[0, \pi]$ |
| arctan$(x)$ | "the angle whose tangent is $x$" | $(-\infty, \infty)$ | $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ |

Note: The domain of arcsin and arccos is $[-1, 1]$ because sine and cosine only ever output values in $[-1, 1]$—so those are the only valid inputs to their inverses. The endpoints $\pm\frac{\pi}{2}$ are **included** in the range of arcsin but **excluded** from the range of arctan, since tan never actually reaches $\pm\infty$.

Example: State the domain and range of $f(x) =$ arcsin$(2x)$.
The argument $2x$ must satisfy $-1 \leq 2x \leq 1$, so $-\dfrac{1}{2} \leq x \leq \dfrac{1}{2}$.
Domain: $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Example: State the domain and range of $f(x) =$ arctan$(x - 1)$.
arctan accepts all real numbers, so $x - 1$ places no restriction on $x$.
Domain: $(-\infty, \infty)$, Range: $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

## Reading domain and range from a graph

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**Tips for reading graphs:**

- Closed circle • means the endpoint **is included**
- Open circle ∘ means the endpoint **is not included**
- For domain: look at horizontal extent (left to right)
- For range: look at vertical extent (bottom to top)

**Practice problems.** Find the domain and range of each function. Write answers in interval notation.

**1.** $f(x) = 5x - 2$

**2.** $f(x) = x^2 - 4$

**3.** $f(x) = -x^2 + 9$

**4.** $f(x) = \sqrt{x+5}$

**5.** $f(x) = \sqrt{3-x}$

**6.** $f(x) = \dfrac{1}{x+4}$

**7.** See the graph below for $f(x) = \dfrac{2}{x^2-1}$

**8.** $f(x) = e^x + 1$

**9.** $f(x) = \ln(x-4)$

**10.** $f(x) = (x-3)^2 - 1$

**11.** $f(x) = \sin(x) + 2$

**12.** $f(x) = 3\cos(x)$

**13.** $f(x) = \arcsin(x)$

**14.** $f(x) = \arccos(x)$

**15.** $f(x) = \arctan(x)$

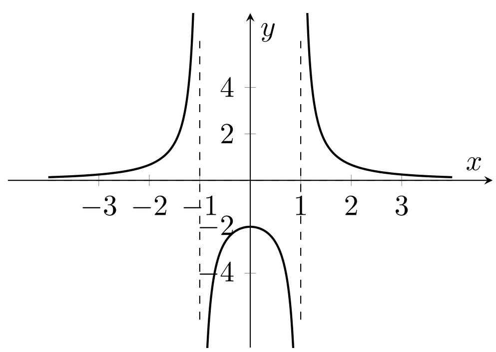
**16.** $f(x) = \arcsin(x-1)$

**17.** $f(x) = \arccos(3x)$

**18.** $f(x) = \arctan(x+2)$

**19.** $f(x) = \arcsin\left(\dfrac{x}{2}\right)$

**20.** $f(x) = 2\arctan(x) - 1$




$$f(x) = \frac{2}{x^2-1}$$

**Answers:**

1. D: $(-\infty,\infty)$, R: $(-\infty,\infty)$ 2. D: $(-\infty,\infty)$, R: $[-4,\infty)$ 3. D: $(-\infty,\infty)$, R: $(-\infty,9]$ 4. D: $[-5,\infty)$, R: $[0,\infty)$
5. D: $(-\infty,3]$, R: $[0,\infty)$ 6. D: $(-\infty,-4)\cup(-4,\infty)$, R: $(-\infty,0)\cup(0,\infty)$ 7. D: $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$, R: $(-\infty,-2]\cup(0,\infty)$
8. D: $(-\infty,\infty)$, R: $(1,\infty)$ 9. D: $(4,\infty)$, R: $(-\infty,\infty)$ 10. D: $(-\infty,\infty)$, R: $[-1,\infty)$ 11. D: $(-\infty,\infty)$, R: $[1,3]$
12. D: $(-\infty,\infty)$, R: $[-3,3]$ 13. D: $[-1,1]$, R: $[-\frac{\pi}{2},\frac{\pi}{2}]$ 14. D: $[-1,1]$, R: $[0,\pi]$ 15. D: $(-\infty,\infty)$, R: $(-\frac{\pi}{2},\frac{\pi}{2})$
16. D: $[0,2]$, R: $[-\frac{\pi}{2},\frac{\pi}{2}]$ 17. D: $[-\frac{1}{3},\frac{1}{3}]$, R: $[0,\pi]$ 18. D: $(-\infty,\infty)$, R: $(-\frac{\pi}{2},\frac{\pi}{2})$ 19. D: $[-2,2]$, R: $[-\frac{\pi}{2},\frac{\pi}{2}]$
20. D: $(-\infty,\infty)$, R: $(-\pi-1,\pi-1)$

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