Quick Review Of Differentiation and Integration From Calculus I and II

In this review we will cover:

- Differentiation rules and examples
- Integration of indefinite and definite integrals
- Improper integrals
- Integration by substitution and integration by parts

Differentiation

Derivatives provide us with the rate of change of a function.

General differentiation rules you need to know:

Derivative of a constant: $\frac{d}{dx} c = 0$

Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$

Derivative of a constant times a function: $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$ or simply $(cf)' = cf'$

keep the constant as is and multiply by the derivative of a function

Product Rule: $(f \cdot g)' = f' \cdot g + g' \cdot f$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$

Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ or $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Differentiate the outer function keeping the inner function the same and multiply by the derivatives of the inner function

Some function-specific derivatives:

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

Exercise 1: Find the derivatives of

a) $f(x) = x^4 \sin 2x$

b) $g(x) = \dfrac{\cos x}{\ln x}$

c) $f(x) = 5e^{x^2} + \sqrt{x} - \dfrac{2}{x} + 3$

d) $h(x) = \ln(\tan(e^{3x}))$

Integration

a) Indefinite integral is a function (also called antiderivative), written as

$\int f(x)dx = F(x) + C$ where $F'(x) = f(x)$

For example: $\int x^3\, dx = \dfrac{x^4}{4} + C$

b) Definite integral $\int_a^b f(x)\,dx = F(b) - F(a)$ is the area under $y = f(x)$ from $x = a$ to $x = b$

where $F'(x) = f(x)$ and f is continous on $[a, b]$.

For example: $\int_1^5 \dfrac{2}{x^2}\,dx = -\dfrac{2}{x}\Big|_1^5 = -\dfrac{2}{5} - \left(-\dfrac{2}{1}\right) = \dfrac{8}{5}$

Some basic integration formulas you may need:

$$\int k\,dx = kx + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$$

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)dx = -\csc(x) + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

Exercise 2: Determine the following indefinite integrals.

Two rules you must know:

$$\int c f(x)\,dx = c\int f(x)\,dx$$

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

a) $\int (4x^8 - 2 + 5x^{-2/3})\,dx$

b) $\int (x^3 + 5)^2\,dx$

c) $\int 3e^{-x}\,dx$

d) $\int \frac{2x^{11} + 5x^{-2} + x^3}{x^3}\,dx$

e) $\int 3x\left(x^7 - \frac{4}{x^3}\right)dx$

f) $\int \frac{1}{2x+3}\,dx$

g) $\int (\sin(2x) - 4\cos(5x))\,dx$

h) $\int (3x+5)^6\,dx$

i) $\int \left(\frac{1}{\sec x} + \csc^2 x\right)dx$

j) $\int \frac{e^{3x} - 2e^{2x} + 3e^{-x}}{e^{2x}}\,dx$

Exercise 3: Evaluate the following definite integrals.

a) $\int_{-1}^{10} (50x - 6x^2)\, dx$

b) $\int_{4}^{16} \frac{x - \sqrt{x}}{x^3}\, dx$

c) $\int_{0}^{3} 2e^t (e^{0.25t} + 1)\, dt$

Improper integrals

There are two kinds of improper integrals:

a) Integrals with infinite limits of integration

For example:

$$\int_0^{\infty} e^{-2t}\, dt = \lim_{b \to \infty} \int_0^b e^{-2t}\, dt = \lim_{b \to \infty} -\frac{1}{2} e^{-2t} \Big|_0^b = \lim_{b \to \infty} -\frac{1}{2} e^{-2b} - \left(-\frac{1}{2} e^{-2(0)}\right) = 0 + \frac{1}{2} = \frac{1}{2}$$

b) Integrals with discontinuous integrands

For example:

$$\int_0^1 \frac{1}{x^2}\, dx = \lim_{b \to 0^+} \int_b^1 \frac{1}{x^2}\, dx = \lim_{b \to 0^+} -\frac{1}{x} \Big|_b^1 = \lim_{b \to 0^+} \left[-\frac{1}{1} - \left(-\frac{1}{b}\right) \right] = \infty$$

There are also improper integrals that require splitting and using two independent limits:

$$\int_{-\infty}^{\infty} e^{-2t}\, dt = \int_{-\infty}^{0} e^{-2t}\, dt + \int_0^{\infty} e^{-2t}\, dt = \lim_{a \to \infty} \int_a^0 e^{-2t}\, dt + \lim_{b \to \infty} \int_0^b e^{-2t}\, dt$$

Exercise 4: Evaluate each integral.

a) $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

b) $\int_2^{\infty} \frac{1}{x}\, dx$

c) $\int_0^{\infty} -3e^{-3x}\, dx$

Not all integrals are as simple as the examples we've seen so far. What if we have to evaluate the following integrals?

$$\int x^2\sqrt{x^3+5}\,dx \qquad \int \frac{x}{(20-x^2)^5}\,dx \qquad \int x^2\sin(2x^3)\,dx$$

For such integrals, we need to do the chain rule in reverse, also known as the substitution method.

Integration by substitution

This method can be used whenever the integrand function is recognized to be of the form $f(g(x))\cdot g'(x)$

The following guideline summarizes the steps involved in u-substitution method:

Integration by Substitution

Choose a new variable u. Usually try choosing u to be some inner function of the integrand whose derivative is also in the integrand.

Compute du.

Replace all terms in the original integrand with expressions involving u and du.

Evaluate the resulting u integrand. (If you can't, you may need to try a different u or a different method of integration.)

Replace u with the corresponding expression in x.

For example, to evaluate $\int \frac{x}{x^2+2}\,dx$, choose $u = x^2+2$ since its derivative is $2x$ and we can write the entire expression in terms of u and du.

$$u = x^2 + 2$$

$$du = 2x\,dx \longrightarrow \frac{du}{2} = x\,dx$$

When substituting we should get the entire expression in terms of u and du.

$$\int \frac{x}{x^2+2}\,dx = \int \frac{\frac{du}{2}}{u} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+2) + C$$

Exercise 5: Evaluate the following integrals:

a) $\int 2x(x^2+4)^{10}\,dx$

b) $\int x^2\sqrt{x^3+5}\,dx$

c) $\int \frac{x}{(20-x^2)^5}\,dx$

d) $\int x^2 \sin(2x^3)\,dx$

e) $\int \frac{3x\,dx}{\sqrt{4x^2+5}}$

f) $\int xe^{x^2}\,dx$

Less apparent substitution

Exercise 6: Evaluate

a) $\int x^2\sqrt{x-1}\,dx$

b) $\int \frac{x\,dx}{\sqrt{2x+1}}$

c) $\int \frac{x}{x-2}\,dx$

Substitution for definite integrals

Method 1: Use u-substitution to find an antiderivative, (i.e. solve the indefinite integral) and then use the Fundamental Theorem to evaluate $F(b) - F(a)$

Method 2: Change the limits of integration as you change variables from x to u. This method tends to be more efficient because you do not need to change the function of u back to function of x.

Exercise 7: Evaluate

a) $\int_0^4 \frac{x}{x^2+1}\,dx$

b) $\int_0^1 (x+1)\sqrt{2x+x^2}\,dx$

c) $\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$

If the integration by substitution does not work and the integral consists of a product of functions (or contains inverse trig functions or logarithmic functions), try integration by parts.

Integration by parts

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

To choose u, you may use **LIATE** which tells you the order of preference for choosing u. Each letter represents a type of function (**L**-logarithmic, **I**-inverse trig, **A**-arithmetic, **T**-trigonometric, **E**-exponential).

Find du by differentiating u and find v by integrating dv.

Substitute u, dv, v, du in the above formula.

If the resulting integral in the second term cannot be evaluated, see if you need to do integration by parts again or if any other method can be used for that term.

For example, to evaluate $\int x\sin 3x\,dx$

$u = x \qquad dv = \sin 3x\,dx$

$du = dx \qquad v = -\frac{1}{3}\cos 3x$

Applying the formula,

$$\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\left(\frac{1}{3}\sin 3x\right) + C$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

Exercise 8: Evaluate

a) $\int x^2 \ln x\,dx$

b) $\int (x^2+1)e^{2x}\,dx$

c) $\int x\sec^2 x\,dx$

d) $\int \sin^{-1} x\,dx$

Integration by parts for definite integrals

For definite integrals that require integration by parts, use the following formula:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

For example, to evaluate $\int_0^1 te^t\,dt$

$u = t \qquad dv = e^t\,dt$

$du = dt \qquad v = e^t$

Applying the formula,

$$\int_0^1 te^t\,dt = te^t\Big|_0^1 - \int_0^1 e^t\,dt = te^t\Big|_0^1 - e^t\Big|_0^1 = (1e^1 - 0e^0) - (e^1 - e^0) = e - e + 1 = 1$$

Exercise 9: Evaluate

a) $\int_{1/2}^{e/2} \ln 2x\,dx$

b) $\int_0^1 xe^{3x}\,dx$

c) $\int_0^{\pi} 2x\cos x\,dx$

ANSWERS:

1. a) $4x^3\sin 2x + 2x^4\cos 2x$

   b) $\dfrac{-\sin x\ln x - \frac{1}{x}\cos x}{(\ln x)^2}$

   c) $10xe^{x^2} + \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x^2}$

   d) $\dfrac{3e^{3x}\sec^2(e^{3x})}{\tan(e^{3x})}$

2. a) $\dfrac{4}{9}x^9 - 2x + 15x^{1/3} + C$

   b) $\dfrac{1}{7}x^7 + \dfrac{5}{2}x^4 + 25x + C$

   c) $-3e^{-x}$

   d) $\dfrac{2}{9}x^9 - \dfrac{5}{4}x^{-4} + x + C$

   e) $\dfrac{1}{3}x^9 + 12x^{-1} + C$

   f) $\dfrac{1}{2}\ln|2x+3| + C$

   g) $-\dfrac{1}{2}\cos 2x - \dfrac{4}{5}\sin 5x + C$

   h) $\dfrac{(3x+5)^7}{21} + C$

   i) $\sin x - \cot x + C$

   j) $e^x - 2x - e^{-3x} + C$

3. a) 473

   b) 11/96

   c) 104.605

4. a) 2

   b) diverges

   c) - 1

5. a) $\dfrac{(x^2+4)^{11}}{11} + C$

   b) $\dfrac{2}{9}(x^3+5)^{3/2} + C$

   c) $\dfrac{1}{8(20-x^2)^4} + C$

   d) $-\dfrac{1}{6}\cos(2x^3) + C$

   e) $\dfrac{3}{4}\sqrt{4x^2+5} + C$

   f) $\dfrac{e^{x^2}}{2} + C$

6. a) $\dfrac{2}{7}(x-1)^{7/2} + \dfrac{4}{5}(x-1)^{5/2} + \dfrac{2}{3}(x-1)^{3/2} + C$

   b) $\dfrac{1}{6}(2x+1)^{3/2} - \dfrac{1}{2}(2x+1)^{1/2} + C$

   c) $x + 2\ln|x-2| + C$

7. a) $\dfrac{\ln(17)}{2}$

   b) $\sqrt{3}$

   c) 1/5

8. a) $\dfrac{x^3\ln x}{3} - \dfrac{1}{9}x^3 + C$

   b) $\dfrac{x^2e^{2x}}{2} - \dfrac{xe^{2x}}{2} + \dfrac{3e^{2x}}{4} + C$

   c) $x\tan x + \ln|\cos x| + C$

   d) $x\sin^{-1}x + \sqrt{1-x^2} + C$

9. a) 1/2

   b) $\dfrac{1}{9} + \dfrac{2e^3}{9}$

   c) -4

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Call: 905.721.8668 ext. 6578
Email: studentlearning@ontariotechu.ca
Website: ontariotechu.ca/studentlearning

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