Derivative by first principles

The definition of the derivative

The derivative of a function $f(x)$ at a point x is defined as:

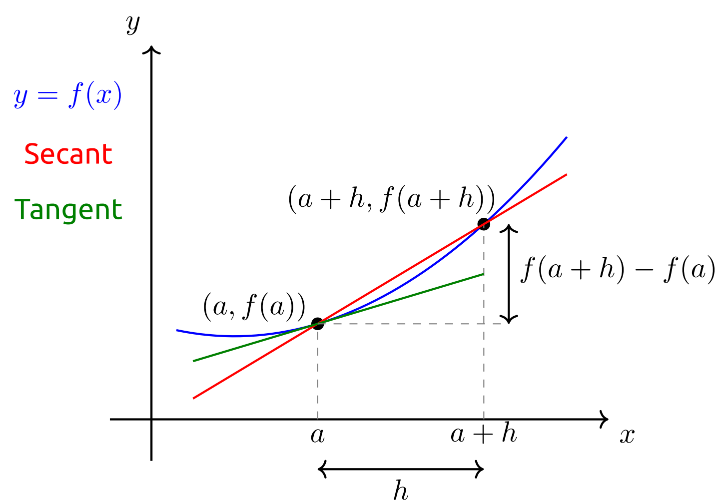
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is also called the **difference quotient** definition or finding the derivative **from first principles**.
An equivalent form using a as the point of interest:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Geometric interpretation

The derivative $f'(a)$ represents the **slope of the tangent line** to the curve $y = f(x)$ at the point $(a, f(a))$.



The secant line through $(a, f(a))$ and $(a+h, f(a+h))$ has slope $\frac{f(a+h) - f(a)}{h}$.
As $h \to 0$, the secant line approaches the tangent line, and the slope approaches $f'(a)$.

Strategy for finding derivatives by first principles

Step 1: Write out the difference quotient $\frac{f(x+h) - f(x)}{h}$
Step 2: Substitute $f(x+h)$ by replacing every x in $f(x)$ with $(x+h)$

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Step 3: Expand and simplify the numerator
Step 4: Factor out h from the numerator and cancel with the denominator
Step 5: Take the limit as $h \to 0$

Example: Linear function

Example: Find the derivative of $f(x) = 3x + 5$ using first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)+5] - [3x+5]}{h} \\ &= \lim_{h\to 0} \frac{3x + 3h + 5 - 3x - 5}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3 \end{aligned}$$

Therefore, $f'(x) = 3$.

Example: Quadratic function

Example: Find the derivative of $f(x) = x^2$ using first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) = 2x \end{aligned}$$

Therefore, $f'(x) = 2x$.

Example: Cubic function

Example: Find the derivative of $f(x) = x^3$ using first principles.
Recall the expansion: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}$$

Therefore, $f'(x) = 3x^2$.

Example: Square root function

Example: Find the derivative of $f(x) = \sqrt{x}$ using first principles.
For this example, we use the **conjugate method** to rationalize the numerator.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

Therefore, $f'(x) = \dfrac{1}{2\sqrt{x}}$.

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Example: Rational function

Example: Find the derivative of $f(x) = \dfrac{1}{x}$ using first principles.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{x-(x+h)}{x(x+h)} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{x(x+h)} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} \\ &= \frac{-1}{x \cdot x} \\ &= -\frac{1}{x^2} \end{aligned}$$

Therefore, $f'(x) = -\dfrac{1}{x^2}$.

Common algebraic techniques

Function type	**Technique**
Polynomial (x^n)	Expand using binomial theorem, cancel h
Square root ($\sqrt{x}$)	Multiply by conjugate to rationalize
Rational ($\frac{1}{x}$)	Find common denominator, simplify
Trigonometric	Use trig identities and special limits

Caution: You cannot simply "plug in $h = 0$" at the start—this gives $\frac{0}{0}$, which is indeterminate. You must algebraically simplify first to cancel the h in the denominator.