Derivatives

Definition and Interpretation

If $y = f(x)$ then the derivative of $f(x)$ with respect to x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It represents the slope of a tangent, the instantaneous rate of change of a function.

Notation

If $y = f(x)$, these are all acceptable ways to write the derivative of this function:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x)$$

If the derivative of $y = f(x)$ is evaluated at $x = a$, the notation to show this is any of the following:

$$f'(a) = y'|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = Df(a)$$

Higher Order

The second derivative is obtained by taking the derivative of the first derivative and is denoted by $f''(x) = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2}$.

In general, higher order nth derivative is denoted as $f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^nf}{dx^n}$.

Derivative Rules

If $f(x)$ and $g(x)$ are differentiable, and c and n are any real numbers,

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}[cf(x)] = cf'(x) \qquad \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$ **Product Rule:** $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ **Chain Rule:** $\frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)g'(x)$

For more information or to book an appointment

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Common Derivatives

Exponential and Logarithmic Functions

$\frac{d}{dx}(e^x) = e^x$ $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\frac{d}{dx}(a^x) = a^x \ln a$ $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Trigonometric Functions

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$

Inverse Trigonometric Functions

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Hyperbolic Functions

$\frac{d}{dx}(\sinh x) = \cosh x$ $\frac{d}{dx}(\cosh x) = \sinh x$ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$ $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Inverse Hyperbolic Functions

$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$ $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$ $\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$ $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

Implicit Differentiation

The implicit differentiation is used when the given equation cannot be solved for y such that $y = f(x)$. For example, $y^2x + \cos y = 3$. When differentiating implicitly, one needs to keep in mind that the derivative of y is y' or $\frac{dy}{dx}$. Here are the steps:

Step 1: Differentiate both sides of the equation with respect to x.

Step 2: Solve for $\frac{dy}{dx}$.

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Using Derivatives to Identify Shape of the Graph

If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.

If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.

If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.

If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Critical points: $x = c$ is a critical point of $f(x)$ provided that $f'(c) = 0$ or $f'(c)$ doesn't exist.

Inflection points: $x = c$ is an inflection point of $f(x)$ if the concavity changes at $x = c$. It is found by analyzing first where $f''(x) = 0$ or $f''(x)$ doesn't exist.

Linear Approximation

The equation of the tangent line at $x = a$, which can be used to approximate a function $f(x)$ near $x = a$, is given by

$$L(x) = f(a) + f'(a)(x - a)$$

Partial Derivatives

If a function has two or more independent variables, the partial derivatives are found by holding all other variables constant and only taking the derivative with respect to a given variable.

For example, if $z = f(x, y)$, the partial derivative of z with respect to x is found by keeping y constant and taking the derivative with respect to x. The notation used is any one of the following:

$$f_x(x, y) = \frac{\partial f}{dx} = \frac{\partial z}{\partial x}$$

Likewise, for the partial with respect to y, where x is constant and the derivative is taken with respect to y, the notation is any of the following:

$$f_y(x, y) = \frac{\partial f}{dy} = \frac{\partial z}{\partial y}$$

For **higher order partial derivatives**, such as f_{yxx}, first find f_y, then f_{yx} by differentiating f_y while keeping y constant, and finally f_{yxx}.

The notation used is $f_{yxx} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right) = \frac{\partial^3 f}{\partial^2 x \partial y}$.

Chain Rule: If $z = f(x, y)$ is a differentiable function of x and y, but x and y are functions of one variable t, i.e. $x = g(t)$ and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Note: "d" is used instead of "∂" with the function of only one independent variable.

In the case if $z = f(x, y)$ is a differentiable function of x and y, but x and y are functions of two variable t and s, i.e. $x = g(t, s)$ and $y = h(t, s)$, then

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Gradient Vector

The gradient vector consists of partial derivatives. For example, the gradient vector of a function in 2 variables is

$$\nabla f(x, y) = < f_x, f_y > = \begin{bmatrix} f_x \\ f_y \end{bmatrix}.$$

The gradient vector of a function in 3 variables is

$$\nabla f(x, y, z) =< f_x, f_y, f_z > = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}.$$

The gradient gives the direction of the maximum rate of change and is always perpendicular to the contour lines of f.

Directional Derivatives

Let $z = f(x, y)$ be a function, (a, b) a point in the domain and $\hat{u}$ a unit vector. The directional derivative at a point (a, b) in the direction $\hat{u}$ is:

$$D_{\hat{u}} f(a, b) = \nabla f(a, b) \cdot \hat{u}$$