Epsilon-delta definition of a limit

The formal definition

We say that $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

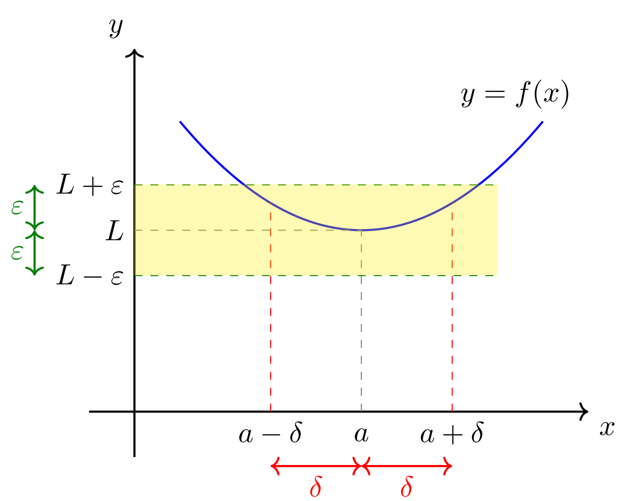
if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

In plain language: We can make $f(x)$ as close to L as we want (within ε) by taking x sufficiently close to a (within δ).

Understanding the notation

Expression	Meaning
$\|x - a\| < \delta$	x is within δ units of a (i.e., $a - \delta < x < a + \delta$)
$0 < \|x - a\|$	x is not equal to a (we exclude the point itself)
$\|f(x) - L\| < \varepsilon$	$f(x)$ is within ε units of L (i.e., $L - \varepsilon < f(x) < L + \varepsilon$)
ε (epsilon)	The "error tolerance" for the output (how close $f(x)$ must be to L)
δ (delta)	The "input tolerance" (how close x must be to a)

Graphical interpretation



The curve must stay within the yellow "box" (between $L - \varepsilon$ and $L + \varepsilon$) whenever x is within δ of a.

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studentlearning@ontariotechu.ca
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Strategy for epsilon-delta proofs

Step 1: Start with what you want to prove: $|f(x) - L| < \varepsilon$
Step 2: Manipulate algebraically to get an expression involving $|x - a|$
Step 3: Determine what δ should be (usually in terms of ε)
Step 4: Write the formal proof, working from $|x - a| < \delta$ to $|f(x) - L| < \varepsilon$

Examples

Example: Find δ in terms of ε so that $\lim_{x \to 3}(2x + 1) = 7$.
We need $|f(x) - L| < \varepsilon$, i.e., $|(2x + 1) - 7| < \varepsilon$.

$$\begin{aligned} |(2x+1) - 7| &< \varepsilon \\ |2x - 6| &< \varepsilon \\ 2|x - 3| &< \varepsilon \\ |x - 3| &< \frac{\varepsilon}{2} \end{aligned}$$

So we should choose $\delta = \dfrac{\varepsilon}{2}$.

Real-world interpretation: Manufacturing tolerances

The epsilon-delta definition mirrors how tolerances work in manufacturing. Consider a machine that produces cylindrical bolts.
Example: A bolt's diameter d (in mm) depends on the lathe's rotation speed s (in RPM) according to the function $d(s) = 0.004s + 6$. The target diameter is 10 mm, achieved when $s = 1000$ RPM. A customer requires bolts with diameter within **0.02 mm** of the target (this is ε). How precisely must the machine's speed be controlled?
Setup: We need $|d(s) - 10| < 0.02$.

$$\begin{aligned} |d(s) - 10| &< 0.02 \\ |(0.004s + 6) - 10| &< 0.02 \\ |0.004s - 4| &< 0.02 \\ 0.004|s - 1000| &< 0.02 \\ |s - 1000| &< 5 \end{aligned}$$

Interpretation: The machine speed must be kept within **5 RPM** of 1000 RPM (this is δ) to guarantee the bolt diameter stays within 0.02 mm of 10 mm.
This is exactly what the epsilon-delta definition captures: to achieve output precision (ε), we need sufficient input precision (δ).

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