Complex power series

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

which converges for $|z - z_0| < R$.
A **Taylor series** is a specific example of a power series.

Taylor series

If f is analytic at z_0 on the disc $|z - z_0| < R$, then the Taylor series for f about z_0 is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

Note: The Maclaurin series is the special case $z_0 = 0$.

Series to know

Finding a function's series representation efficiently usually involves manipulating a known series.

Geometric series: $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (\text{converges for } |z| < 1)$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad (\text{converge for all } z)$$

Laurent series

Laurent series are power series with both positive and negative powers.
Let $f(z)$ be analytic on the annulus $r < |z - z_0| < R$.
Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

converging on the annulus, where $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_\rho^+(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \qquad n \in \mathbb{Z}, \qquad r < \rho < R.$

Note: $C_\rho^+(z_0)$ denotes a positively oriented circle centred at z_0 with radius ρ.

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To find a Laurent series, do *not* use the definition above. Instead, manipulate known Taylor or Maclaurin series into appropriately convergent geometric series.

Example: Find the Laurent series for

$$f(z) = \frac{3z+1}{(z+2)(z-3)}$$

valid on the annulus $2 < |z| < 3$.
Using partial fractions,

$$f(z) = \frac{1}{z+2} + \frac{2}{z-3}.$$

Manipulate the expressions in the form of a convergent geometric series for $|z| > 2$ and $|z| < 3$:

$$\frac{1}{z+2} = \frac{1}{z}\left(\frac{1}{1+\frac{2}{z}}\right) = \frac{1}{z}\left(\frac{1}{1-(-\frac{2}{z})}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n, \quad \text{converges for } \left|-\frac{2}{z}\right| < 1 \text{ (i.e. } 2 < |z|).$$

$$\frac{2}{z-3} = -\frac{2}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = -\frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \quad \text{converges for } \left|\frac{z}{3}\right| < 1 \text{ (i.e. } |z| < 3).$$

Thus,

$$f(z) = \sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n, \qquad 2 < |z| < 3.$$

Note that $f(z)$ has singularities at $z = -2$ and $z = 3$, so there are three regions where $f(z)$ is analytic:

the circle $|z| < 2,$ the annulus $2 < |z| < 3,$ and the unbounded region $|z| > 3.$

Zeros, singularities, poles

Laurent series can be used to classify zeros and singularities.
A function f analytic in $D(z_0)$ has a **zero of order** k at z_0 if

$$f^{(n)}(z_0) = 0 \quad \text{for } n = 0, 1, \ldots, k-1, \qquad f^{(k)}(z_0) \neq 0.$$

Example: Show that $f(z) = \sin z - z$ has a zero of order 3 at $z_0 = 0$.
Using the Taylor expansion,

$$\sin z - z = -\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$$

so the leading term is $-z^3/3!$, and the zero has order 3.

$f(z) = \dfrac{1}{z+3}$ has an isolated singularity at $z_0 = -3$.

$f(z) = \log z$ has a non-isolated singularity at $z_0 = 0$, which lies on a branch cut.

Let f have an isolated singularity at z_0 with Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Such a singularity can be classified as follows.

- Removable singularity: $a_n = 0 \quad$ for all $n < 0$.

  Example:

  $$f_1(z) = \frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

  f_1 has a removable singularity at $z = 0$.

- Essential singularity: $a_n \neq 0 \quad$ for infinitely many $n < 0$.

  Example:

  $$f_2(z) = \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \cdots$$

  f_2 has an essential singularity at $z = 0$.

- Pole of order k: $a_{-k} \neq 0 \quad$ and $\quad a_{-n} = 0$ for $n > k$.

  - The order of the pole is the **most negative power** in the Laurent series.

  Example:

  $$f_3(z) = \frac{\cos z}{z^3} = \frac{1}{z^3} - \frac{1}{2!z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots$$

  f_3 has a pole of order 3 at $z = 0$.

Useful corollaries

If $f(z) = \dfrac{h(z)}{(z - z_0)^k}$, with h analytic and $h(z_0) \neq 0$, then f has a pole of order k at z_0.

If z_0 is a zero of order k of $f(z)$, then z_0 is a pole of order k of $1/f(z)$.

If z_0 is a pole of order m of $f(z)$ and of order n of $g(z)$, then z_0 has a pole of order $m + n$ of $f(z)g(z)$, and $f(z)/g(z)$ has the following behaviour:

$$\begin{cases} \text{If } m = n, & z_0 \text{ is a removable singularity of } \dfrac{f(z)}{g(z)}, \\ \text{If } m > n, & z_0 \text{ is a pole of order } m - n \text{ of } \dfrac{f(z)}{g(z)}, \\ \text{If } m < n, & z_0 \text{ is a zero of order } n - m \text{ of } \dfrac{f(z)}{g(z)}. \end{cases}$$

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