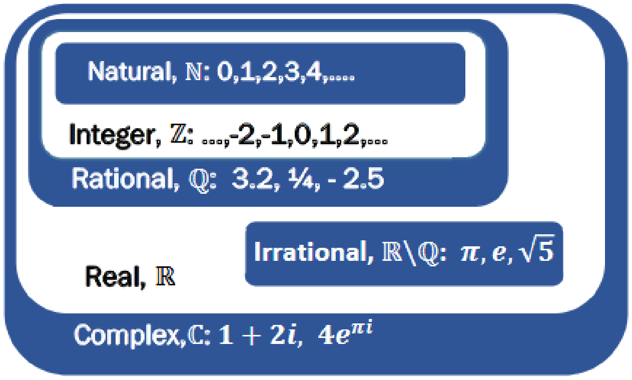
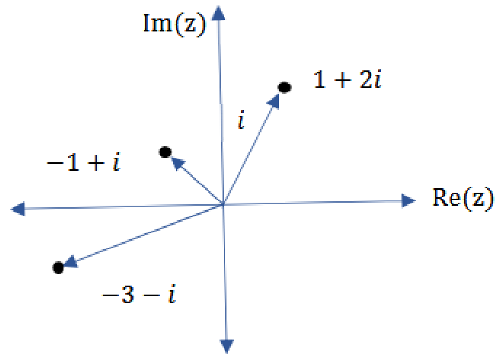
Complex numbers

The complex number system is applied to solve and simplify many math problems ranging from trigonometry and geometry to algebra and calculus.





Cartesian form

$$z = x + iy = \text{Re}(z) + i\,\text{Im}(z)$$

Addition/Subtraction: Add real and imaginary parts.

$z_1 \pm z_2 = \text{Re}(z_1) \pm \text{Re}(z_2) + i(\text{Im}(z_1) \pm \text{Im}(z_2))$

Example: $(-1+i) + (1+2i) = (-1+1) + (1+2)i = 3i$

Multiplication: Use the distributive property and $i^2 = -1$.

Example: $(-1+i)(1+2i) = -1 - 2i + i + 2i^2 = -1 - i - 2 = -3 - i$

Division: Rationalize (multiply by the conjugate).

Example: $\dfrac{1+i}{1+2i} = \dfrac{1+i}{1+2i} \cdot \dfrac{1-2i}{1-2i} = \dfrac{1-i-2i^2}{1+4} = \dfrac{3}{5} - \dfrac{1}{5}i$

Useful Results:

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

$$z\overline{z} = |z|^2$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

$$\frac{1}{i} = -i$$

Polar form

$$z = re^{i\theta} = |z|e^{i\,\text{Arg}(z)}$$

Modulus (notion of length, or distance from zero): $|z| = \sqrt{x^2 + y^2}$

Principle Argument: $\text{Arg}(z) = \theta$ where $\theta \in (-\pi, \pi]$

This can be solved using $\tan(\theta) = \frac{y}{x}$. Remember to take into consideration the signs of both x and y to determine the correct quadrant for θ.

Argument: $\arg(z) = \{\theta + 2\pi k : k \in \mathbb{Z}\}$

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In practice, use whichever complex form is more convenient. Addition and subtraction are easier in Cartesian form whereas multiplication and division are easier in polar form.

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Example: Convert $-1+i$ to polar form.

$$|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$
$$\text{Arg}(z) = \frac{3\pi}{4}$$
$$-1+i = \sqrt{2}e^{\frac{3i\pi}{4}}$$

Note: $\arctan(-1) = -\frac{\pi}{4}$ but the signs of x and y indicate the angle is in Quadrant II so the corresponding angle is $\frac{3\pi}{4}$.

Euler's identity

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$

Example: Convert $2e^{\frac{5i\pi}{6}}$ into Cartesian form.
Using Euler's identity,

$$2e^{\frac{5i\pi}{6}} = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\sqrt{3}+i$$

Powers and roots

De Moivre's Formula:

$$z^n = r^n e^{ni\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

Roots are a bit more complicated since there are multiple solutions that need to be obtained:

$$z^{\frac{1}{n}} = |z|^{\frac{1}{n}} e^{\frac{i(\theta+2k\pi)}{n}} \qquad \text{for } k = 0, 1, 2, ..., n-1$$

Example: Find the cube roots of $1+i\sqrt{3}$.
First convert to polar form: $1+\sqrt{3} = 2e^{\frac{i\pi}{3}}$.
Then $(1+\sqrt{3})^{\frac{1}{3}} = 2^{\frac{1}{3}} e^{i(\frac{\pi}{9}+\frac{2k\pi}{3})}$ for $k = 0, 1, 2$.
So three cube roots of $1+i\sqrt{3}$ are $2^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)$, $2^{\frac{1}{3}}\left(\cos\left(\frac{7\pi}{9}\right) + i\sin\left(\frac{7\pi}{9}\right)\right)$, and $2^{\frac{1}{3}}\left(\cos\left(\frac{13\pi}{9}\right) + i\sin\left(\frac{13\pi}{9}\right)\right)$.