Complex functions

A complex function can be written in terms of its real and imaginary parts:

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y) \quad \text{or} \quad w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$

Exponential function

Power series:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

Some similarities with its real counterpart include:

$$e^0 = 1, \qquad e^z \neq 0, \qquad \frac{d}{dz} e^z = e^z, \qquad e^{z_1} e^{z_2} = e^{z_1 + z_2}.$$

Some notable differences: e^z is not one-to-one, e^z can attain negative values.
A key property used for computating is $e^z = e^x(\cos(y) + i\sin(y))$, hence e^z is periodic:

$$\begin{cases} e^z \equiv 1 & \text{if } z = 2k\pi i, \quad k \in \mathbb{Z} \\ e^{z_1} \equiv e^{z_2} & \text{if } z_1 = z_2 + 2k\pi i, \quad k \in \mathbb{Z} \end{cases}$$

Example: $e^{2+\frac{\pi i}{4}} = e^2 \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = e^2 \left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$

Example: Find all $z \in \mathbb{C}$ such that $e^z = \sqrt{3} - i$.
First, write in polar form: $|\sqrt{3} - i| = \sqrt{3+1} = 2$ and $\text{Arg}(\sqrt{3} - i) = -\frac{\pi}{6}$.
Then, $e^z = e^{x+iy} = e^x e^{iy} = 2e^{-\frac{\pi i}{6}}$ only when $e^x = 2$ and $e^{iy} = e^{-\frac{i\pi}{6}}$ or $e^{iy - (-i\frac{\pi}{6})} = 1$.
This is true when $x = \ln(2)$ and $y + \frac{\pi}{6} = 2\pi k$ or $y = \frac{\pi}{6}(12k - 1)$ for $k \in \mathbb{Z}$.

Notation: $f(z) = e^{\frac{1}{3}}$ may be interpreted as any of the complex third roots so sometimes $\exp\{z\}$ is used to denote that a single value is chosen, i.e. $\exp\left(\frac{1}{3}\right) \approx 1.3956$ whereas

$$e^{1/3} \approx 1.3956 \left(\cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}\right), \quad k = 0, 1, 2.$$

Logarithmic function

Multivalued logarithm:

$$\log(z) = \{\ln|z| + i(\text{Arg}(z) + 2n\pi)\} = \{\ln(z) + i\arg(z)\} \qquad \text{for } n \in \mathbb{Z} \text{ and } z \neq 0$$

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Single-valued logarithm (principal value):

$$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z) \qquad \text{for } z \neq 0$$

Example: $\text{Log}(\sqrt{3} - i) = \ln(2) + i(-\frac{\pi}{6})$.

Properties:
$\exp(\text{Log}(z)) = z$ for $z \neq 0$ and $\text{Log}(\exp(z)) = z$ for $-\pi < \text{Im}\{z\} \leq \pi$.
It also gives a way to define complex powers, for $\alpha \in \mathbb{C}$:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\ln|z| + i \arg(z))}$$

Note: if α is rational, only a finite number of z^α result.

Example: Determine all possible values of $i^{\frac{\sqrt{3}}{4}}$.
First, if $z = i$ then $|z| = \sqrt{0^2 + 1^2} = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$.

$$\begin{aligned} z^\alpha &= \left\{ e^{\frac{\sqrt{3}}{4}(\ln(1) + i(\frac{\pi}{2} + 2\pi n))} : n \in \mathbb{Z} \right\} \\ &= \left\{ e^{\frac{\sqrt{3}}{4}(i(\frac{\pi}{2} + 2\pi n))} : n \in \mathbb{Z} \right\} = \left\{ e^{i\frac{\sqrt{3}\pi}{8}(1+4n)} : n \in \mathbb{Z} \right\} \\ &= \left\{ \cos\left(\frac{\sqrt{3}\pi}{8}(1+4n)\right) + i \sin\left(\frac{\sqrt{3}\pi}{8}(1+4n)\right) : n \in \mathbb{Z} \right\} \end{aligned}$$

Example: Determine all possible values of $(i + \sqrt{3})^i$.
Ans: $\left\{ e^{-\frac{\pi}{6} + 2\pi n} (\cos(\ln 2) + i \sin(\ln 2)) : n \in \mathbb{Z} \right\}$

Note: The Maple command `evalc` yields the principal value.

Trigonometric functions

The familiar identities and derivatives for trigonometric functions still hold in $\mathbb{C}$.

$$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \cos(z) = \frac{1}{2i}(e^{iz} + e^{-iz})$$

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) \qquad \cos(x + iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sinh(z) = \frac{1}{2}(e^z - e^{-z}) \qquad \cosh(z) = \frac{1}{2}(e^z + e^{-z})$$

A notable difference: $\sin(z)$ and $\cos(z)$ are unbounded, unlike $|\sin(x)| \leq 1$ for $x \in \mathbb{R}$.

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \qquad |\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

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