Complex power series

A power series is a series of the form: $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ which converge for $|z-z_0| < R$

A Taylor Series is a specific example of power series.

Taylor series

If f that is analytic at z_0, on the circle $|z-z_0| < R$, the Taylor Series for f around z_0 is:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + f'(z_0)(z-z_0) + f''(z_0)(z-z_0)^2 + \cdots$$

Note: the Maclaurin Series is the special case where $z_0 = 0$.

Series to know:

Finding a function's series representation efficiently involves the manipulation of a known series.

$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ (geometric series)	$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$	$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$	$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$
converges for $\|z\| < 1$	converges for all z		

Laurent series

Laurent Series are power series with positive and negative powers.

Let $f(z)$ be analytic on the annulus $r < |z-z_0| < R$. Then,

$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converging on the annulus.

where $a_{-n} = \frac{1}{2\pi i}\int_{C_\rho^+(z_0)} \frac{f(z)}{(z-z_0)^{-n+1}}dz$ and $a_n = \frac{1}{2\pi i}\int_{C_\rho^+(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}dz$

Note: $C_\rho^+(z_0)$ is a positively oriented circle with centre z_0 and radius ρ where $r < \rho < R$.

To find a Laurent Series, don't use the definition above. Instead, use known Taylor/Maclaurin series or manipulate rational expressions into the appropriately convergent geometric series.

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Example: Find the Laurent Series for $f(z) = \frac{3z+1}{(z+2)(z-3)}$ valid for the annulus $2 < |z| < 3$.

Using partial fraction decomposition, $f(z) = \frac{3z+1}{(z+2)(z-3)} = \frac{1}{z+2} + \frac{2}{z-3}$

Manipulate the expressions in the form of a geometric series convergent for $2 < |z|$ and $|z| < 3$:

$\frac{1}{z+2} = \frac{1}{z}\left(\frac{1}{1+\frac{2}{z}}\right) = \frac{1}{z}\left(\frac{1}{1-\left(\frac{-2}{z}\right)}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n$ converges for $\left|-\frac{2}{z}\right| < 1$ (ie. $2 < |z|$)

$\frac{2}{z-3} = \frac{-1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$ converges for $\left|\frac{z}{3}\right| < 1$ (ie. $|z| < 3$)

Putting this together,

$$f(z) = \frac{3z+1}{(z+2)(z-3)} = \frac{1}{z+2} + \frac{2}{z-3} = \sum_{n=1}^{\infty}\frac{(-2)^{n-1}}{z^n} + \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

Note that $f(z)$ has singularities at $z = -2$ and $z = 3$, so there are 3 regions where $f(z)$ is analytic:

the circle $|z| < 2$, the annulus $2 < |z| < 3$, and the unbounded region $|z| > 3$.

Show the Laurent Series for $f(z)$ valid on $|z| < 2$ is the Maclaurin Series $f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^{n+1}}$.

Show the Laurent Series for $f(z)$ valid on $|z| > 3$ is $f(z) = \sum_{n=1}^{\infty}\frac{(-2)^{n-1}+(3)^{n-1}}{z^n}$.

Zeros, singularities, poles

We can use the Laurent series expansion to quantify zeros and singularities.

A function f analytic in $D(z_0)$ that has a zero of order k at z_0 if

$$f^{(n)}(z_0) = 0 \text{ for } n = 0,1,..k-1, \text{ but } f^{(k)} \neq 0$$

Example: Show $f(z) = \sin(z) - z$ has a zero of order 3 at $z_0 = 0$.

$f(0) = \sin(0) - 0 = 0$

$f'(0) = \cos(0) - 1 = 1 - 1 = 0$

$f''(z_0) = -\sin(0) = 0$

$f'''(z_0) = -\cos(0) = -1 \neq 0$

You can classify more quickly if you use a function's Taylor expansion (it has a zero of order k if the first k Taylor coefficients vanish):

$f(z) = \sin(z) - z = \frac{-z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

Looking at the leading term, one can conclude that $f(z)$ has a zero of order 3 at $z_0 = 0$.

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$f(z) = \frac{1}{z+3}$ has an isolated singularity at $z_0 = -3$.

$f(z) = \log(z)$ has a non-isolated singularity at $z_0 = 0$, one point along a branch cut of singularities.

Let f have an isolated singularity at z_0 with a Laurent series of $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$.

It can be classified as a:

- **Removable singularity:** $a_n = 0$ for all $n < 0$

  Example: $f_1 = \frac{e^z-1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$ $\quad$ f_1 has a removable discontinuity at 0

- **Essential singularity:** $a_{-n} \neq 0$ for infinitely many n

  Example: $f_2 = \cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \cdots$ $\quad$ f_2 has an essential singularity at 0

- **Pole of order k:** $a_{-k} \neq 0$ and $a_{-n} = 0$ for n > k (k singular terms in Laurent series)
  - i.e. the order of the pole is the most negative power

  Example: $f_3 = \frac{\cos(z)}{z^3} = \frac{1}{z^3} - \frac{1}{2!z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots$ $\quad$ f_3 has a pole of order 3 at 0

Useful corollaries:

If $f(z) = \frac{h(z)}{(z-z_0)^k}$ with h analytic and $h(z_0) \neq 0$, then f has a pole of order k at z_0.

If z_0 is a zero of order k of $f(z)$, then z_0 is a pole of order k of $\frac{1}{f(z)}$.

If z_0 has a pole of order m of $f(z)$ and of order n of $g(z)$, then z_0 has a pole of order $m + n$ of $f(z)g(z)$ and $f(z)/g(z)$ has the following behaviour:

If $m > n$ or $m = n$, then z_0 is a removable singularity of $f(z)/g(z)$
If $m < n$, then z_0 is a pole of order $m - n$ of $f(z)/g(z)$