



Complex integration

If $f(t) = u(t) + iv(t)$, where t is a real variable, we can extend our idea of integration easily as follows:

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i \int_a^b v(t) dt$$

Example: $\int_0^1 (2t + i)^3 dt = \int_i^{2+i} \frac{1}{2} u^3 du = \frac{(2+i)^4}{8} - \frac{i^4}{8} = \frac{(2+i)^4 + 1}{8}$

Contour integrals

In order to integrate a complex mapping $f(z)$ from a to b we need to have a specified curve to follow.

Rules for integration

$$\int_C af(z)dz = a \int_C f(z)dz$$

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

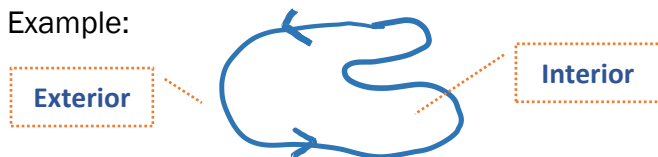
$$\int_{C+K} f(z)dz = \int_C f(z)dz + \int_K f(z)dz$$

$$\int_{-C} f(z)dz = - \int_C f(z)dz$$

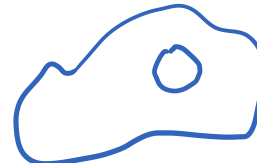
Important terms to know

Simply connected: any simple (does not cross itself) closed contour without holes.

Example:



Non-Example:



If the contour, C , is positively orientated that means the interior is on the left and exterior on the right.

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Important contours to know

A line segment from $z = z_0$ to $z = z_1$:

$$z(t) = z_0 + t(z_1 - z_0), 0 \leq t \leq 1$$

A circle of radius $R > 0$, centre at z_0 :

$$z(t) = z_0 + Re^{it}, \theta_0 \leq t \leq \theta_0 + 2\pi \quad (\text{counter-clockwise})$$

$$z(t) = z_0 + Re^{i(2\pi + \theta_0 - t)}, \theta_0 \leq t \leq \theta_0 + 2\pi \quad (\text{clockwise})$$

Examples (other parametrizations of the same curve are possible):

$$z(t) = i + 2e^{i\pi t}, 0 \leq t \leq 1 \quad \text{a semi-circle, radius 2, centre } i, \text{ counterclockwise}$$

$$z(t) = t + i, -1 \leq t \leq 1 \quad \text{a line segment from } -1 + i \text{ to } 1 + i$$

$$z(t) = 4 + e^{i(2\pi - t)}, 0 \leq t \leq 2\pi \quad \text{a circle, radius 1, centre 4, clockwise}$$

If $f(z)$ is defined on a set including the contour, C , integrating along the contour can be done as follows:

$$\int_C f(z) dz = \int_a^b f(z(t)) z'(t) dt, a \leq t \leq b$$

Example: Compute $\int_C \bar{z} dz$ on the contour shown.

First, parameterize the contour. $C: z(t) = 2e^{i(\pi-t)}, 0 \leq t \leq \pi$

Using Euler's Identity this can be rewritten as:

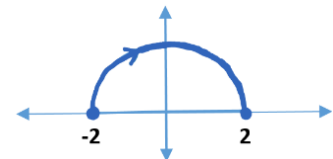
$$2e^{i(\pi-t)} = 2 \cos(\pi - t) + 2i \sin(\pi - t) = -2 \cos(t) + 2i \sin(t)$$

$$f(z(t)) = \overline{-2 \cos(t) + 2i \sin(t)} = 2 \cos(\pi - t) - 2i \sin(\pi - t) = -2 \cos(t) - 2i \sin(t)$$

$$z'(t) = 2 \sin(t) + 2i \cos(t)$$

$$\begin{aligned} \int_C \bar{z} dz &= \int_0^\pi (-2 \cos(t) - 2i \sin(t))(2 \sin(t) + 2i \cos(t)) dt = \int_0^\pi -4i(\sin^2 t + \cos^2 t) dt \\ &= \int_0^\pi -4i dt = -4i\pi \end{aligned}$$

Note that the function \bar{z} is not analytic, a different contour with the same initial and terminal points could yield a different result.



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Fundamental theorem

Fundamental Theorem of Integration: If f is **analytic** in a simply connected domain, D , and z_0 and z_1 are any two points joined by contour C lying entirely in D , then

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

In this situation, the function is independent of the path connecting the initial and terminal point.

Example: Compute $\int_C \cos(z) dz$ where C has the parametrization $z(t) = e^{i\pi t}$, $-\frac{1}{2} \leq t \leq \frac{1}{2}$.

$$\int_C \cos(z) dz = \int_{-i}^i \cos(z) dz = \sin(i) - \sin(-i) = \frac{-i}{2}(e^{-1} - e) + \frac{i}{2}(e - e^{-1}) = -ie^{-1} + ie$$

Simplified by recalling from Euler's Identity: $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

ML- Inequality

If $f(z)$ is continuous on the contour C then,

$$\left| \int_C f(z)dz \right| \leq ML \text{ where } |f(z)| \leq M \text{ on } C \text{ and } L \text{ is the length of } C$$

Example:

If $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ establish a bound for $\left| \int_C f(z)dz \right|$.

$$|f(z)| \leq \frac{1}{r} \text{ and } L = 2\pi r$$

$$\left| \int_C f(z)dz \right| \leq 2\pi$$

Integrals over simple, closed contours

Cauchy-Goursat theorem

If f is analytic in a simply connected domain, D , and C is a simple, closed contour that lies in D , then,

$$\int_C f(z)dz = 0$$

Example: $\int_C \frac{1}{z+1} dz = 0$ for any contour that does not include $z = -1$ on its interior or on C .

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How do we deal with more challenging integrals, where the antiderivative isn't known?

Cauchy's integral formula

If f is analytic in a simply connected domain, D , and C is a simple, closed contour that lies in D , and z_0 is any point inside C , then,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

And for derivatives,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n = 1, 2, 3, \dots$$

Example: Compute $\int_C \frac{e^{i\pi z}}{(4z-1)(z+2)} dz$ over the positively oriented contour $C: |z| = 1$

Notice that only the singularity $z = \frac{1}{4}$ is in the contour as $z = -2$ is outside the circle of radius 1. Identify the remainder of the integrand as $f(z)$ to apply Cauchy's Integral formula.

$$\begin{aligned} \int_C \frac{e^{i2\pi z}}{(4z-1)(z+2)} dz &= \int_C \frac{e^{i2\pi z}}{4\left(z-\frac{1}{4}\right)(z-(-2))} dz \\ &= \int_C \frac{\frac{e^{i2\pi z}}{4(z+2)}}{\left(z-\frac{1}{4}\right)} dz = \int_C \frac{f(z)}{\left(z-\frac{1}{4}\right)} dz = 2\pi i f\left(\frac{1}{4}\right) = 2\pi \frac{e^{\frac{i\pi}{2}}}{-7} = \frac{2\pi\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)}{-7} = 2\pi i \end{aligned}$$

Example: Compute $\int_C \frac{\cos(3z)}{z^3} dz$ over the positively oriented ellipse $C: \frac{x^2}{9} + \frac{y^2}{4} = 0$

Here $z_0 = 0$, a singularity that lies inside the contour, so using the theorem:

$$\int_C \frac{\cos(3z)}{z^3} dz = \frac{2\pi i f^{(2)}(0)}{2!} = -9\pi i$$

Cauchy's Integral Formula provided us with a "trick" to integrate, allowing these examples to be completed without the need to parameterize the contour.

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