Complex functions

A complex function can be written in terms of its real and imaginary parts:

$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$ or $w = f(z) = f\left(re^{i\theta}\right) = u(r, \theta) + i\, v(r, \theta)$

Exponential function

Power series: $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$

Some similarities with its real counterpart include: $e^0 = 1$, $e^z \neq 0$, $\frac{d}{dz} e^z = e^z$, $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

Some notable differences: e^z is not one-to-one, e^z can attain negative values

A key property used for computation is $e^z = e^x(\cos y + i \sin y)$, hence e^z is periodic:

$$\mathrm{e}^{z} \equiv 1 \qquad \text{if } z = 2k\pi\, i, k \in Z$$

$$\mathrm{e}^{z_1} \equiv \mathrm{e}^{z_2} \qquad \text{if } z_1 = z_2 + 2k\pi\, i,\ k \in Z$$

Example: $\mathrm{e}^{2+\mathrm{i}\frac{\pi}{4}} = \mathrm{e}^2(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \mathrm{e}^2\left(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}\right)$

Example: Find all $z \in C$ such that $e^z = \sqrt{3} - i$

First, write in polar form: $\left|\sqrt{3} - i\right| = \sqrt{3 + 1} = 2$ and $Arg\left(\sqrt{3} - i\right) = -\frac{\pi}{6}$

Then, $e^z = e^{x+iy} = e^x e^{iy} = 2e^{-i\frac{\pi}{6}}$ only when:

$e^x = 2$ and $e^{iy} = e^{-\frac{i\pi}{6}}$ or $e^{iy - \left(-\frac{\pi}{6}\right)} = 1$

Which is true when $x = \ln(2)$ and $y + \frac{\pi}{6} = 2\pi k$ or $y = \frac{\pi}{6}(12k - 1), k \in Z$

Notation: $f(z) = e^{\frac{1}{3}}$ may be interpreted as any of the complex third roots so sometimes exp(z) is used to denote that a single value is chosen, i.e. $\exp\left(\frac{1}{3}\right) \approx 1.3956$ whereas $e^{1/3} \approx 1.3956\left(\cos\left(\frac{2\pi k}{3}\right) + isin\left(\frac{2\pi k}{3}\right)\right)$ for $k = 0, 1, 2.$

Logarithmic function

Multivalued logarithm:

$\log(z) = \{\ln|z| + i(Arg(z) + 2n\pi)\} = \{\ln(z) + iarg(z)\}$, where n is an integer and for $z \neq 0$

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Single-valued logarithm (principal value):

$\text{Log}(z) = \ln|z| + iArg(z)$, for $z \neq 0$,

Example: $\text{Log}(\sqrt{3} - \text{i}) = \ln(2) + i\left(-\frac{\pi}{6}\right)$

Properties:

$\boldsymbol{exp(Log(z)) = z}$ **for** $\boldsymbol{z \neq 0}$ & $\boldsymbol{Log(exp(z)) = z}$ **for** $\boldsymbol{-\pi < Im(z) \leq \pi}$ *(Recall: Arg(z)* $\in (-\pi, \pi]$*)*

It also gives a way to define complex powers, for $\alpha \in C$:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\ln|z| + \mathrm{i}\arg(z))}$$

Note: if α is rational, only a finite number of z^α result

Example: Determine all possible values of $(\text{i})^{\sqrt{3}/4}$.

First, if $z = i$, then $|z| = \sqrt{0^2 + 1^2} = 1$ and $Arg(z) = \frac{\pi}{2}$

$$z^\alpha = \{ e^{\frac{\sqrt{3}}{4}\left(\ln(1) + i\left(\frac{\pi}{2} + 2\pi n\right)\right)} |\ n \in Z\}$$

$$= \{ e^{\frac{\sqrt{3}}{4}\left(i\left(\frac{\pi}{2} + 2\pi n\right)\right)} |\ n \in Z\} = \{ e^{i\frac{3\pi}{8}(1+4n)} |\ n \in Z\}$$

$$= \left\{\cos\left(\frac{\sqrt{3}\pi}{8}(1 + 4n)\right) + i\sin\left(\frac{\sqrt{3}\,\pi}{8}(1 + 4n)\right) \middle|\ n \in Z\right\}$$

Example: Determine all possible values of $\left(i + \sqrt{3}\right)^i$.

Ans: $\left\{\text{e}^{-\frac{\pi}{6} + 2\pi n}[\cos(\ln 2) + i\sin(\ln 2)] \,\middle|\, n \in Z\right\}$

Note: The Maple command **evalc** yields the principal value.

Trigonometric functions

The familiar identities and derivatives for trigonometric functions still hold in C.

$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$, $\cos z = \frac{1}{2i}\left(e^{iz} + e^{-iz}\right)$

$\sin(\text{x} + \text{iy}) = \sin x \cosh y + \text{i}\cos x \sinh y$, $\cos(\text{x} + \text{iy}) = \cos x \cosh y - \text{i}\sin x \sinh y$

$\sinh z = \frac{1}{2}(e^z - e^{-z})$, $\cosh z = \frac{1}{2}(e^z + e^{-z})$

A notable difference: $\sin z$ and $\cos z$ are unbounded, unlike $|\sin x| \leq 1, x \in R$.

$|\sin z|^2 = \sin^2 x + \sinh^2 y$, $|\cos z|^2 = \cos^2 x + \sinh^2 y$