Chi-square tests

Overview

Chi-square (χ^2) tests are used to analyze categorical data. There are two main types:

Test	Purpose
Goodness of Fit	Does the data fit a specified distribution?
Test of Independence	Are two categorical variables independent?

The chi-square test statistic

For both tests, the test statistic is:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where O = observed frequency and E = expected frequency.
Interpretation: Large values of χ^2 indicate a poor fit between observed and expected values, leading to rejection of H_0.

Goodness of fit test

Use when: Testing whether observed data follows a hypothesized distribution.
Hypotheses:

- H_0: The data follows the specified distribution
- H_a: The data does not follow the specified distribution

Degrees of freedom: $df = k - 1$ where k is the number of categories.
Example: A die is rolled 120 times with the following results. Is the die fair?

Outcome	1	2	3	4	5	6
Observed (O)	25	17	15	23	24	16
Expected (E)	20	20	20	20	20	20

If the die is fair, each outcome has probability $\frac{1}{6}$, so $E = 120 \times \frac{1}{6} = 20$ for each.

$$\chi^2 = \frac{(25-20)^2}{20} + \frac{(17-20)^2}{20} + \frac{(15-20)^2}{20} + \frac{(23-20)^2}{20} + \frac{(24-20)^2}{20} + \frac{(16-20)^2}{20}$$
$$= \frac{25+9+25+9+16+16}{20} = \frac{100}{20} = 5.0$$

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Degrees of freedom: $df = 6 - 1 = 5$
Critical value at $\alpha = 0.05$: $\chi^2_{0.05,5} = 11.07$
Since $\chi^2 = 5.0 < 11.07$, we fail to reject H_0. There is not enough evidence to conclude the die is unfair.

Test of independence

Use when: Testing whether two categorical variables are independent (no association).
Hypotheses:

- H_0: The variables are independent
- H_a: The variables are not independent (they are associated)

Expected frequencies: For each cell in the table:

$$E = \frac{(\text{row total})(\text{column total})}{\text{grand total}}$$

Degrees of freedom: $df = (r-1)(c-1)$ where r = rows, c = columns.
Example: A survey asks 400 people about exercise habits and stress levels. Is there an association?
Observed counts:

	Low Stress	Medium Stress	High Stress	**Row Total**
Exercises regularly	60	50	30	140
Does not exercise	40	80	140	260
Column Total	100	130	170	400

Calculate expected frequencies:

$$E_{11} = \frac{140 \times 100}{400} = 35 \qquad E_{12} = \frac{140 \times 130}{400} = 45.5 \qquad E_{13} = \frac{140 \times 170}{400} = 59.5$$
$$E_{21} = \frac{260 \times 100}{400} = 65 \qquad E_{22} = \frac{260 \times 130}{400} = 84.5 \qquad E_{23} = \frac{260 \times 170}{400} = 110.5$$

Expected counts:

	Low Stress	Medium Stress	High Stress
Exercises regularly	35	45.5	59.5
Does not exercise	65	84.5	110.5

Calculate χ^2:

$$\chi^2 = \frac{(60-35)^2}{35} + \frac{(50-45.5)^2}{45.5} + \frac{(30-59.5)^2}{59.5} + \frac{(40-65)^2}{65} + \frac{(80-84.5)^2}{84.5} + \frac{(140-110.5)^2}{110.5}$$
$$= \frac{625}{35} + \frac{20.25}{45.5} + \frac{870.25}{59.5} + \frac{625}{65} + \frac{20.25}{84.5} + \frac{870.25}{110.5}$$
$$= 17.86 + 0.45 + 14.63 + 9.62 + 0.24 + 7.88 = 50.68$$

Degrees of freedom: $df = (2-1)(3-1) = 2$
Critical value at $\alpha = 0.05$: $\chi^2_{0.05,2} = 5.99$
Since $\chi^2 = 50.68 > 5.99$, we reject H_0. There is significant evidence of an association between exercise and stress levels.

Conditions for chi-square tests

- Data must be counts (frequencies), not percentages
- Observations must be independent
- Expected frequency in each cell should be at least 5
- If expected frequencies are too small, combine categories

Chi-square critical values (right-tail)

df	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$
1	2.706	3.841	5.024	6.635
2	4.605	5.991	7.378	9.210
3	6.251	7.815	9.348	11.345
4	7.779	9.488	11.143	13.277
5	9.236	11.070	12.833	15.086
6	10.645	12.592	14.449	16.812
7	12.017	14.067	16.013	18.475
8	13.362	15.507	17.535	20.090
9	14.684	16.919	19.023	21.666
10	15.987	18.307	20.483	23.209

Decision rule: Reject H_0 if $\chi^2 > \chi^2_{\alpha,df}$ (critical value from table).
Note: Chi-square tests are always right-tailed because we're looking for large deviations from expected values.