Bessel Functions

Bessel functions are a family of special functions that arise as solutions to certain differential equations. They appear in many areas of mathematics, physics, and engineering, particularly in problems involving wave propagation, heat conduction, vibrations, and systems with circular or cylindrical symmetry.

Definition and Differential Equation

Bessel functions arise as solutions to **Bessel's differential equation**, a second-order ordinary differential equation given by:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

where $\nu \geq 0$ is a real constant representing the order of the Bessel function.

The two standard linearly independent solutions of this Bessel's differential equation are the **Bessel functions of the first kind**, denoted by $J_\nu(x)$, and the **Bessel functions of the second kind**, denoted by $Y_\nu(x)$.
The general solution depends on whether the order ν is an integer:
If ν is an integer,

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

If ν is not an integer,

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x),$$

since $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent.

The Bessel function of the first kind, $J_\nu(x)$, remains finite at $x = 0$ for $\nu \geq 0$, whereas the Bessel function of the second kind, $Y_\nu(x)$, is singular at the origin:

$$\lim_{x \to 0^+} Y_\nu(x) = -\infty.$$

Series Representations (Order $\nu \geq 0$)

Because the origin $x = 0$ is a **regular singular point** of Bessel's differential equation, standard power series methods fail. Instead, the standard canonical expressions are derived using the Frobenius method by substituting a generalized series solution of the form:

$$y(x) = \sum_{m=0}^{\infty} c_m x^{m+r}$$

Student Learning Support, Teaching and Learning Centre
studentlearning@ontariotechu.ca
ontariotechu.ca/studentlearning

Solving the resulting indicial equation yields the roots $r = \pm\nu$. Substituting these roots back into the algebraic recurrence relations produces the explicit series definitions below:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}$$

where $\Gamma(z)$ is the Gamma function. If $\nu = n$ is a non-negative integer, $\Gamma(m+n+1) = (m+n)!$:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left(\frac{x}{2}\right)^{2m+n}$$

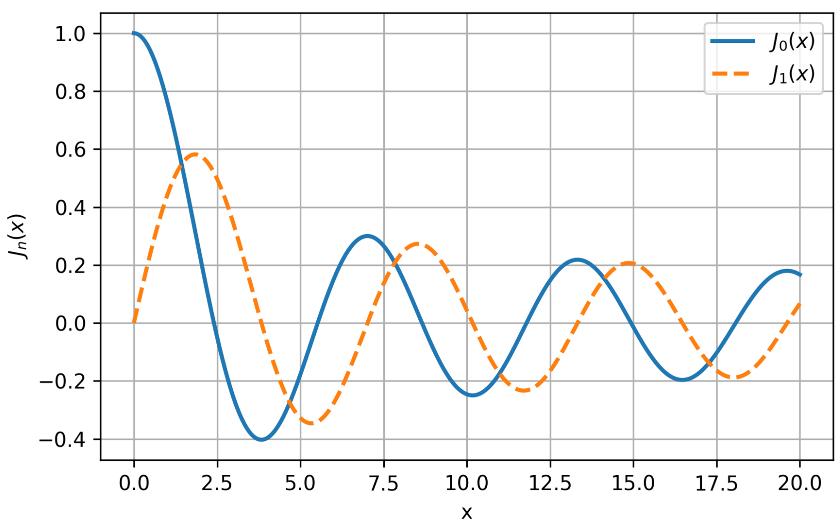
For non-integer orders, the second kind is defined in terms of the first kind by

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)},$$

with the integer order $Y_n(x)$ defined as the limit as $\nu \to n$.

Visualizing Bessel Functions

Bessel functions oscillate in a way that resembles sine and cosine, but there are important differences. Their amplitudes gradually decrease as x increases.



Recognizing and Solving Parametric Bessel Equations

Many differential equations do not initially appear in standard form but can be transformed into Bessel equations. A highly useful generalized differential equation format is:

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0$$

Student Learning Support, Teaching and Learning Centre
studentlearning@ontariotechu.ca
ontariotechu.ca/studentlearning

Its general solution is explicitly given by the formula:

$$y = x^a \left[c_1 J_p(bx^c) + c_2 Y_p(bx^c)\right]$$

Matching Strategy:

1. Divide the entire differential equation by the coefficient of y'' to put it into the standard form $y'' + P(x)y' + Q(x)y = 0$.
2. Equate the coefficient of y' to $\dfrac{1-2a}{x}$ and solve for a.
3. Look at the first term inside the parentheses of the y coefficient ($b^2c^2x^{2c-2}$). Determine the power exponent to solve for c, then solve for b.
4. Use the remaining constant term over x^2 to solve for the order p.

Example: Find the general solution of $x^2y'' + 2xy' + xy = 0$ in terms of Bessel functions.
Step 1. Convert to standard form: Divide through by x^2:

$$y'' + \frac{2}{x}y' + \frac{1}{x}y = 0$$

Step 2. Match the coefficients:

- Find a: $\dfrac{1-2a}{x} = \dfrac{2}{x} \implies 1-2a = 2 \implies a = -\frac{1}{2}$
- Find c and b: Match $\dfrac{1}{x} = x^{-1}$ with the term $b^2c^2x^{2c-2}$.

$$2c - 2 = -1 \implies 2c = 1 \implies c = \frac{1}{2}$$

$$b^2c^2 = b^2\left(\frac{1}{2}\right)^2 = 1 \implies \frac{b^2}{4} = 1 \implies b = 2$$

- Find p: There is no $\dfrac{1}{x^2}$ term in our equation, so its coefficient is 0.

$$a^2 - p^2c^2 = 0 \implies \left(-\frac{1}{2}\right)^2 - p^2\left(\frac{1}{2}\right)^2 = 0 \implies \frac{1}{4} = \frac{1}{4}p^2 \implies p = 1$$

Step 3. Construct the solution: Plug $a = -\frac{1}{2}$, $b = 2$, $c = \frac{1}{2}$, and $p = 1$ into the master solution form:

$$y = x^{-1/2}\left[c_1 J_1(2x^{1/2}) + c_2 Y_1(2x^{1/2})\right]$$

Evaluating Bessel Functions

Most Bessel function values are not calculated by hand. Instead, they are found using tools like MATLAB, Python, Maple, and Mathematica or tables provided in textbooks or formula sheets.

Example: For an input of $x = 2$, the value of the zero-order Bessel function of the first kind is approximately:

$$J_0(2) \approx 0.224$$

Zeros of Bessel Functions

A zero of a Bessel function is a value of x where the function crosses the axis:

$$J_n(x) = 0$$

These roots are uniquely important across physical applications because they directly determine natural frequencies of vibrating circular membranes, eigenvalues and resonance modes in wave propagation, and allowable solutions to cylindrical boundary-value problems.
For the zero-order function $J_0(x)$, the first three positive zeros are approximately:

$$2.405, \qquad 5.520, \qquad 8.654$$

These specific numbers appear frequently in engineering and physics applications.

Worked Example: Boundary Value Problem

Suppose the radial solution to a differential equation is modeled by:

$$R(r) = J_0(\lambda r)$$

and the system boundary condition requires the field to vanish at the boundary radius $r = 1$:

$$R(1) = 0$$

Step 1. Substitute the boundary condition: Set $r = 1$ in the solution:

$$J_0(\lambda \cdot 1) = 0 \implies J_0(\lambda) = 0$$

Step 2. Solve for the parameter λ: For a non-trivial solution, λ must be a root (zero) of the J_0 function. The smallest positive zero corresponding to the system's lowest energy configuration yields:

$$\boxed{\lambda \approx 2.405}$$

Key Terminology: The smallest positive root is often called the **fundamental mode** of the system.

Recurrence Relations

Bessel functions satisfy several universal recurrence identities. These equations apply identically to $J_\nu(x)$, $Y_\nu(x)$, and any linear combination $Z_\nu(x) = c_1 J_\nu(x) + c_2 Y_\nu(x)$.

Differential Form	**Algebraic Form**
$\dfrac{d}{dx}\left[x^\nu Z_\nu(x)\right] = x^\nu Z_{\nu-1}(x)$	$Z_{\nu-1}(x) + Z_{\nu+1}(x) = \dfrac{2\nu}{x} Z_\nu(x)$
$\dfrac{d}{dx}\left[x^{-\nu} Z_\nu(x)\right] = -x^{-\nu} Z_{\nu+1}(x)$	$Z_{\nu-1}(x) - Z_{\nu+1}(x) = 2Z_\nu'(x)$
$xZ_\nu'(x) = \nu Z_\nu(x) - xZ_{\nu+1}(x)$	$xZ_\nu'(x) = -\nu Z_\nu(x) + xZ_{\nu-1}(x)$

Example: Express $J_3(x)$ entirely in terms of $J_0(x)$ and $J_1(x)$.

Solution: Isolate $Z_{\nu+1}(x)$ from the first algebraic recurrence relation: $J_{\nu+1}(x) = \dfrac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$.

$$\text{For } \nu = 1: \quad J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$
$$\text{For } \nu = 2: \quad J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$$

Substitute $J_2(x)$ into the expression for $J_3(x)$:

$$J_3(x) = \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$$

Tip: Recurrence relations are especially useful when higher-order Bessel functions are required but only $J_0(x)$ and $J_1(x)$ are known or tabulated.

Integral Representations (Integer Order n)

When $\nu = n$ is an integer, $J_n(x)$ can be represented by a definite integral over a finite domain:

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - x \sin\theta)\, d\theta$$

Setting $n = 0$ yields the highly useful zero-order expression:

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta)\, d\theta$$

Modified Bessel Functions

When the variable is imaginary ($x \to ix$), the governing equation turns into the **Modified Bessel Equation**:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2) y = 0$$

The general solution utilizes the real-valued modified Bessel functions: $y(x) = c_1 I_\nu(x) + c_2 K_\nu(x)$.

Function	**Definition**	**Behavior at Origin ($x \to 0$)**
First Kind (I_ν)	$I_\nu(x) = i^{-\nu} J_\nu(ix)$	Bounded ($I_0(0) = 1$, $I_{\nu>0}(0) = 0$)
Second Kind (K_ν)	$K_\nu(x) = \frac{\pi}{2} \frac{I^{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$	Unbounded ($\lim_{x \to 0^+} K_\nu(x) = \infty$)

Half-Integer Orders (Spherical Bessel Functions)

When the order is a half-integer (e.g., $\nu = \pm 1/2$), the infinite Frobenius series terminates into a closed-form combination of elementary trigonometric functions. These are highly useful when solving wave or diffusion equations in **spherical coordinates**.
The fractional-order functions of the first and second kind for $\nu = 1/2$ are explicitly given by:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \qquad \text{and} \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Using these definitions, the second kind function of order $1/2$ evaluates cleanly to:

$$Y_{1/2}(x) = -J_{-1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x$$

Modified Functions of Order 1/2: Similarly, the modified Bessel functions of order $1/2$ collapse into elementary hyperbolic functions, which routinely appear in spherical heat dissipation and radial particle shielding profiles:

$$I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh x \qquad \text{and} \qquad K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

Orthogonality of Bessel Functions

Just as sine and cosine functions are orthogonal in Fourier series, Bessel functions are orthogonal with respect to the weight function x. This property allows functions defined on circular or cylindrical domains to be expanded as **Fourier–Bessel series**, making Bessel functions fundamental in solving boundary-value problems involving heat conduction, wave propagation, vibrations, electromagnetics, and neutron diffusion.
Let $\alpha_{n,m}$ denote the mth positive zero of $J_n(x)$. Then

$$\int_0^R x\, J_n\!\left(\frac{\alpha_{n,i}}{R} x\right) J_n\!\left(\frac{\alpha_{n,j}}{R} x\right) dx = \begin{cases} 0, & i \neq j, \\ \dfrac{R^2}{2} \left[J_{n+1}(\alpha_{n,i})\right]^2, & i = j. \end{cases}$$

Key Idea: Orthogonality allows Bessel functions to play the same role in cylindrical coordinates that sine and cosine functions play in Cartesian coordinates.