Applications of integration

Arc length of a curve

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Strategy:

1. Find $\frac{dy}{dx}$
2. Square it: $\left(\frac{dy}{dx}\right)^2$
3. Add 1 and take the square root: $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
4. Set up and evaluate the integral

Example: Find the arc length of $y = \frac{1}{3}x^3$ from $x = 0$ to $x = 2$.

Solution: First, find the derivative: $\frac{dy}{dx} = x^2$.
Then:

$$L = \int_0^2 \sqrt{1 + (x^2)^2}\, dx = \int_0^2 \sqrt{1 + x^4}\, dx$$

Note: Arc length integrals are often difficult to evaluate by hand. In practice, they are frequently computed numerically or left as "set up only."

Surface area of revolution

If f is positive and has a continuous derivative, the surface area obtained by rotating $y = f(x)$, $a \leq x \leq b$, about the x-axis is:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example: Find the surface area obtained by revolving $y = \frac{1}{3}x^3$ around the x-axis from $x = 0$ to $x = 2$.

Solution:

$$S = \int_0^2 2\pi \cdot \frac{1}{3}x^3 \cdot \sqrt{1 + x^4}\, dx = \frac{2\pi}{3} \int_0^2 x^3 \sqrt{1 + x^4}\, dx$$

Using substitution $u = 1 + x^4$, $du = 4x^3\,dx$:

$$S = \frac{2\pi}{3} \cdot \frac{1}{4} \int_1^{17} u^{1/2}\,du = \frac{\pi}{6} \cdot \frac{2}{3} u^{3/2}\Big|_1^{17} = \frac{\pi}{9}\left(17^{3/2} - 1\right) = \frac{\pi}{9}(17\sqrt{17} - 1)$$

Hydrostatic force on a dam

The force on a thin horizontal plate submerged in fluid is:

$$F = \rho \cdot g \cdot d \cdot A$$

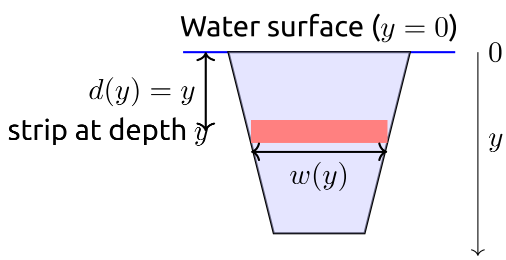
where ρ is fluid density, g is gravitational acceleration, d is depth, and A is area.
Units:

- Metric: $\rho = 1000$ kg/m^3, $g = 9.8$ m/s^2, Force in Newtons (N)
- Imperial: $\rho g = 62.5$ lb/ft^3, Force in pounds (lb)

For a vertical surface (like a dam), pressure increases with depth. We use horizontal strips and integrate:

$$F = \int_a^b \rho \cdot g \cdot d(y) \cdot w(y)\,dy$$

where $d(y)$ is the depth of the strip and $w(y)$ is the width of the strip.
Strategy: Set $y = 0$ at the water surface with y positive downward. Then $d(y) = y$.



Example: A rectangular plate 8 m wide and 3 m tall is submerged vertically, with its top edge 1 m below the water surface. Find the hydrostatic force.
Solution: The plate extends from $y = 1$ to $y = 4$. Width is constant: $w(y) = 8$.

$$\begin{aligned} F &= \int_1^4 (1000)(9.8)(y)(8)\,dy \\ &= 78400 \int_1^4 y\,dy \\ &= 78400 \cdot \frac{y^2}{2}\Big|_1^4 \\ &= 78400 \cdot \frac{16-1}{2} \\ &= 588{,}000 \text{ N} \end{aligned}$$

Cardiac output (Biology application)

Cardiac output is the volume of blood pumped by the heart per unit time. It can be measured using the **dye dilution method**.

If A is the amount of dye injected and $c(t)$ is the concentration of dye at time t, then:

$$F = \frac{A}{\displaystyle\int_0^T c(t)\,dt}$$

where F is the cardiac output and T is the time for the dye to clear.

Example: 5 mg of dye is injected, and the concentration is modelled by $c(t) = \frac{1}{3}t(9-t)$ mg/L for $0 \le t \le 9$ seconds. Find the cardiac output.

Solution: First, evaluate the integral:

$$\begin{aligned}\int_0^9 c(t)\,dt &= \int_0^9 \frac{1}{3}t(9-t)\,dt = \frac{1}{3}\int_0^9 (9t - t^2)\,dt \\ &= \frac{1}{3}\left[\frac{9t^2}{2} - \frac{t^3}{3}\right]_0^9 \\ &= \frac{1}{3}\left[\frac{9(81)}{2} - \frac{729}{3}\right] \\ &= \frac{1}{3}\left[\frac{729}{2} - 243\right] = \frac{1}{3}\cdot\frac{243}{2} = \frac{81}{2}\end{aligned}$$

Therefore:

$$F = \frac{5}{81/2} = \frac{10}{81} \approx 0.123 \text{ L/s} = 7.4 \text{ L/min}$$

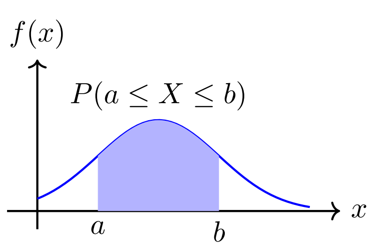
Probability density functions

A **probability density function (pdf)** $f(x)$ for a continuous random variable X satisfies:

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Requirements for a pdf:

1. $f(x) \ge 0$ for all x
2. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$



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Example: Consider $f(x) = \begin{cases} \frac{3}{14}x(x+1) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$

Find $P(X < 1)$.

Solution:

$$P(X<1) = \int_0^1 \frac{3}{14}x(x+1)\,dx = \frac{3}{14}\int_0^1 (x^2+x)\,dx$$
$$= \frac{3}{14}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = \frac{3}{14}\left[\frac{1}{3}+\frac{1}{2}\right] = \frac{3}{14}\cdot\frac{5}{6} = \frac{5}{28}$$

Example: The time t (in days) between major earthquakes follows an exponential distribution with pdf $f(t) = \frac{1}{960}e^{-t/960}$ for $t \ge 0$. Find the probability that the time between earthquakes exceeds 365 days.

Solution:

$$\begin{aligned} P(T>365) &= \int_{365}^{\infty} \frac{1}{960}e^{-t/960}\,dt \\ &= \lim_{b\to\infty}\left[-e^{-t/960}\right]_{365}^{b} \\ &= \lim_{b\to\infty}\left(-e^{-b/960}+e^{-365/960}\right) \\ &= 0 + e^{-365/960} \\ &= e^{-0.380} \approx 0.684 \end{aligned}$$

There is about a 68.4% chance that the time between major earthquakes exceeds one year.