Adjacency matrices

Definition

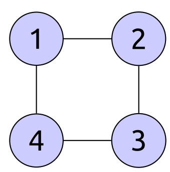
The **adjacency matrix** A of a graph G with n vertices is an $n \times n$ matrix where:

$$A_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is adjacent to vertex } j \\ 0 & \text{otherwise} \end{cases}$$

For simple undirected graphs, the adjacency matrix is:

- **Symmetric**: $A_{ij} = A_{ji}$ (edges go both ways)
- **Zero diagonal**: $A_{ii} = 0$ (no loops)

Example: Graph to matrix



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Reading the matrix:

- Row 1: vertex 1 is adjacent to vertices 2 and 4
- Row 2: vertex 2 is adjacent to vertices 1 and 3
- The sum of row i equals deg(i)

Example: Matrix to graph

Given the adjacency matrix, draw the graph:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Every vertex is adjacent to every other vertex $\Rightarrow$ This is K_4:

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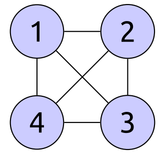


Properties of adjacency matrices

Property	What it tells you
Sum of row i	Degree of vertex i
Sum of all entries	$2\times$ (number of edges)
$A = A^T$	Graph is undirected
All diagonal entries are 0	Graph has no loops
$(A^k)_{ij}$	Number of walks of length k from i to j

Example: For the cycle C_4 above, find the number of walks of length 2 from vertex 1 to vertex 3.
Compute A^2:

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

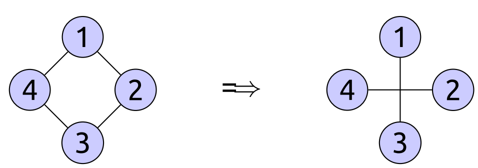
$(A^2)_{13} = 2$, so there are 2 walks of length 2 from vertex 1 to vertex 3: $(1 \to 2 \to 3)$ and $(1 \to 4 \to 3)$.

Graph complement

The **complement** of a graph G, denoted $\overline{G}$, has the same vertices but edges are "flipped": two vertices are adjacent in $\overline{G}$ if and only if they are **not** adjacent in G.
If A is the adjacency matrix of G, then the adjacency matrix of $\overline{G}$ is:

$$\overline{A} = J - I - A$$

where J is the all-ones matrix and I is the identity matrix.
Example:



G (cycle C_4) $\quad\quad$ $\overline{G}$

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Quick reference

Task	Method
Find degree of vertex i	Sum row i (or column i)
Count edges	Sum all entries, divide by 2
Find walks of length k	Compute A^k; entry (i, j) gives count
Find complement matrix	$\overline{A} = J - I - A$
Check isomorphism	Compare vertices, edges, degree sequence first