Residues and residue theorem

Residues definition

Let f have a **nonremovable isolated singularity** at z_0 and the Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. The residue is,

$$\text{Res}[f, z_0] = a_{-1} \text{ where } a_{-1} \text{ is the coefficient of the } \frac{1}{z-z_0} \text{ term of the Laurent series}$$

Example: Find Res$[f, z_0]$ if

$$f(z) = e^{\frac{3}{z-1}}.$$

The Laurent series expansion is

$$e^{\frac{3}{z-1}} = 1 + \frac{3}{z-1} + \frac{3^2}{2!(z-1)^2} + \frac{3^3}{3!(z-1)^3} + \cdots$$

The coefficient of $\frac{1}{z-1}$ is 3, hence

$$\text{Res}[f, 1] = 3.$$

Calculating residues at poles

Expanding the Laurent series to identify the coefficient is a general method that will work to find the residues of essential singularities and poles, but if z_0 is a pole of degree k of $f(z)$ it can be easier to use the following equations to find the residue:

If z_0 is a pole then Res$[f, z_0] = \lim_{z \to z_0} (z - z_0)f(z)$

If z_0 is a pole of order k, then Res$[f, z_0] = \lim_{z \to z_0} \left[\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left(\frac{(z-z_0)^k f(z)}{(k-1)!} \right) \right]$

Try this method to calculate the residue in the previous example.

Cauchy's residue theorem

This simplifies the calculation of contour integrals (over simple closed contours) to just computing residues. It is an extension of the Cauchy integral formula, which helped us evaluate integrands of the form $\frac{f(z)}{(z-z_0)^k}$. It is used to evaluate problems with a finite number of isolated singularities.

If C is a simple, closed, positively-oriented contour and f is analytic inside and on C except at points z_j then,

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \text{Res}[f, z_j]$$

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Example:

Compute $\int_C f(z)\,\mathrm{d}z = \int_C \frac{1-z}{(z-4)(z+2)^3}\,\mathrm{d}z$ over the positively oriented contour $C : |z| = 5$.

The integrand has a simple pole at $z_0 = 4$ and a pole of order 3 at $z_0 = -2$. Both of these points lie within the contour so by Residue Theorem,

$$\text{Res}[f, 4] = \lim_{z\to 4}(z-4)\frac{1-z}{(z-4)(z+2)^3} = \lim_{z\to 4}\frac{1-z}{(z+2)^3} = \frac{-3}{6^3} = \frac{-1}{72}$$

$$\text{Res}[f, -2] = \lim_{z\to -2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{(z+2)^3}{2!}\frac{1-z}{(z-4)(z+2)^3}\right) = \lim_{z\to -2}\frac{1}{(z-4)^2} + \frac{1-z}{(z-4)^3} = \frac{1}{72}$$

$$\int_C \frac{1-z}{(z-4)(z+2)^3}\,\mathrm{d}z = 2\pi i\left(-\frac{1}{72} + \frac{1}{72}\right) = 0$$

Example: Compute $\int_C \frac{\sin(3z)}{z^2(z-1)}\,\mathrm{d}z$ over the positively-oriented ellipse $C : \frac{x^2}{9} + \frac{y^2}{4} = 1$.

The integrand has a pole of order 2 at $z = 0$ and a simple pole at $z = 1$. Both lie inside the contour C, so by the Residue Theorem,

$$\text{Res}[f, 0] = \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^2}{1!}\right)\left(\frac{\sin(3z)}{z^2(z-1)}\right) = \lim_{z\to 0}\left(\frac{3\cos(3z)}{z-1}\right) = -3$$

$$\text{Res}[f, 1] = \lim_{z\to 1}(z-1)\frac{\sin(3z)}{z^2(z-1)} = \sin(3)$$

$$\int_C \frac{\sin(3z)}{z^2(z-1)}\,\mathrm{d}z = 2\pi i(-3 + \sin(3))$$