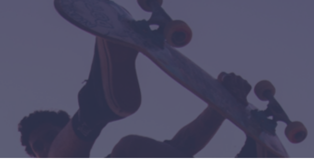
10 Rotation



The Rigid Body Model

A *rigid body* is an extended object whose size and shape don't change. Rigid bodies can't be stretched, compressed, or deformed; all points on the rigid body have the same angular velocity and angular acceleration; and rigid bodies can have translational motion, rotational motion, or a combination of both.

Moment of Inertia

The *moment of inertia* is the rotational equivalent of mass, and can be calculated for N particles of mass m_i by

$$I = \sum_i^N m_i r_i^2,$$

where r_i is the distance from the axis of rotation.

For some objects, you can just look up the moment of inertia:

Shape, axis	I	**Shape, axis**	I
Thin rod, centre	$(1/12)ML^2$	Thin rod, end	$(1/3)ML^2$
Plane, centre	$(1/12)Ma^2$	Plane, edge	$(1/3)Ma^2$
Disk	$(1/2)MR^2$	Hoop	MR^2
Solid sphere	$(2/5)MR^2$	Spherical shell	$(2/3)MR^2$

The parallel axis theorem allows you to find the moment of inertia about an axis parallel to the centre of mass axis:

$$I = I_{\text{cm}} + Md^2,$$

where d is the distance from the centre of mass.

The Cross Product

Two vectors can be multiplied together to get another vector through the *cross-product*:

$$\vec{A} \times \vec{B} = (AB \sin\alpha, \text{direction by the right hand rule}).$$

Torque

Torque is the rotational equivalent of force, and is defined to be

$$\vec{\tau} = \vec{r} \times \vec{F},$$

where $\vec{r}$ is the position of where the force is applied measured from the rotation axis.

Note that *gravitational torque* can be found by treating the object as if all its mass were concentrated at the centre of mass:

$$\tau_{\text{grav}} = -Mgx_{\text{cm}}.$$

Rotational Kinetic Energy

A rigid body object rotating with angular speed ω has a rotational kinetic energy given by

$$K_{\text{rot}} = \frac{1}{2} I \omega^2.$$

Centre of Mass

An unconstrained object of mass M with no net force will rotate about its *centre of mass*: the only point that remains motionless. The centre of mass can be found by

$$x_{\text{cm}} = \frac{1}{M} \int x\, dm \quad \text{and} \quad y_{\text{cm}} = \frac{1}{M} \int y\, dm,$$

or, if the object consists of N particles of mass m_i,

$$x_{\text{cm}} = \frac{1}{M} \sum_i^N m_i x_i \quad \text{and} \quad y_{\text{cm}} = \frac{1}{M} \sum_i^N m_i y_i.$$

Newton's Second Law for Rotation

A net torque causes an angular acceleration given by Newton's second law,

$$\alpha = \frac{\tau_{\text{net}}}{I}.$$

If an object is stationary, both the *net force* and *net torque* are zero. This is called *static equilibrium*:

$$\text{Static equilibrium:} \quad \vec{F}_{\text{net}} = \vec{0} \quad \text{and} \quad \vec{\tau}_{\text{net}} = \vec{0}.$$

Rolling

Rolling consists of both *translational* and *rotational* motion; these motions are linked by the *rolling constraint*,

$$v_{\text{cm}} = R\omega.$$

The rolling of a rigid body can ve described as a translation of the centre of mass plus a rotation about the centre of mass. The total kinetic energy of rolling is

$$K = K_{\text{rot}} + K_{\text{cm}} = \frac{1}{2} I_{\text{cm}} \omega^2 + \frac{1}{2} M v_{\text{cm}}^2.$$

Angular Momentum

The *angular momentum* of a *particle* is defined as

$$\vec{L} = \vec{r} \times \vec{p}.$$

In the case that a *rigid body* is rotating about a fixed axis or an axis of symmetry, its angular momentum is

$$\vec{L} = I\vec{\omega}.$$

We can write Newton's second law for rotation in terms of angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{\text{net}},$$

so that a net torque changes the angular momentum of an object.

If the system is *isolated*, so that $\tau_{\text{net}} = 0$, angular momentum is conserved:

$$\vec{L}_i = \vec{L}_f.$$