# Complex integration

If $f(t) = u(t) + iv(t)$ where $t$ is a real variable, we can extend our idea of integration easily as follows:

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t$$

Example: $\int_0^1 (2t+i)^3\,\mathrm{d}t = \int_i^{2+i} \frac{1}{2}u^3\,\mathrm{d}u = \frac{(2+i)^4}{8} - \frac{i^4}{8} = \frac{(2+i)^4+1}{8}$

## Contour integrals

In order to integrate a complex mapping $f(z)$ from $a$ to $b$, we need to have a specified curve to follow.

### Rules for integration

$$\int_C af(z)\,\mathrm{d}z = a\int_C f(z)\,\mathrm{d}z$$

$$\int_C (f(z)+g(z))\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z + \int_C g(z)\,\mathrm{d}z$$

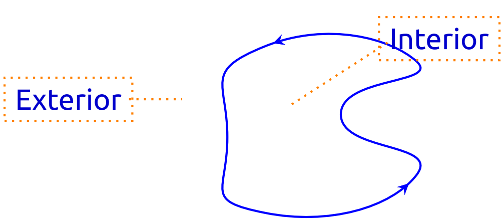
$$\int_{C+K} f(z)\,\mathrm{d}z = \int_C f(z)\,\mathrm{d}z + \int_K f(z)\,\mathrm{d}z$$

$$\int_{-C} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z$$

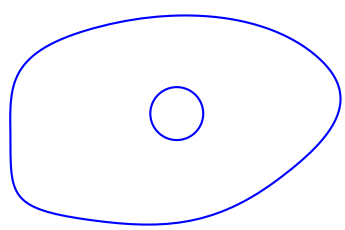
### Important terms to know

Simply connected: any simple (does not cross itself) closed contour without holes.

**Example:**




**Non-Example:**



If the contour $C$ is positively oriented, that means the interior is on the left and exterior is on the right.

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## Important contours to know

A **line segment** from $z = z_0$ to $z = z_1$:

$$z(t) = z_0 + t(z_1 - z_0), \quad 0 \le t \le 1$$

A **circle** of radius $R > 0$, centred at $z_0$:

$$z(t) = z_0 + Re^{it}, \quad \theta_0 \le t \le \theta_0 + 2\pi \qquad \text{(counterclockwise)}$$

$$z(t) = z_0 + Re^{i(2\pi+\theta_0-t)}, \quad \theta_0 \le t \le \theta_0 + 2\pi, \qquad \text{(clockwise)}$$

Examples (other parameterizations of the same curve are possible):

$z(t) = i + 2e^{i\pi t}, 0 \le t \le 1$ — a semi-circle, radius $2$, centre $i$, counterclockwise

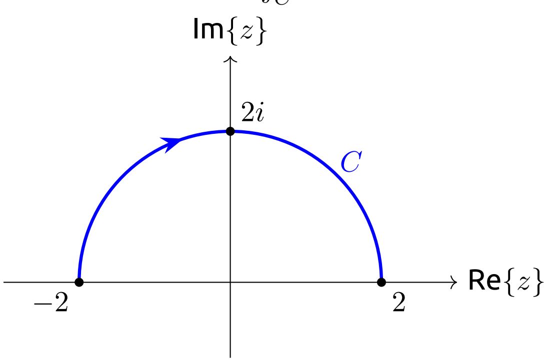
$z(t) = t + i, -1 \le t \le 1$ — a line segment from $-1 + i$ to $1 + i$

$z(t) = 4 + e^{i(2\pi-t)}, 0 \le t \le 2\pi$ — a circle, radius $1$, centre $4$, clockwise

If $f(z)$ is defined on a set including the contour $C$, integrating along the contour can be done as follows:

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t\,, \quad a \le t \le b$$

Example: Compute $\int_C \overline{z}\,\mathrm{d}z$ on the contour shown.




First, parameterize the contour: $C : z(t) = 2e^{i(\pi-t)}, 0 \le t \le \pi$.
Using Euler's identity this can be rewritten as:

$$2e^{i(\pi-t)} = 2\cos(\pi - t) + 2i\sin(\pi - t) = -2\cos(t) + 2i\sin(t)$$

$$f(z(t)) = \overline{-2\cos(t) + 2i\sin(t)} = 2\cos(\pi - t) - 2i\sin(\pi - t) = -2\cos(t) - 2i\sin(t)$$

$$z'(t) = 2\sin(t) + 2i\cos(t)$$

$$\int_C \overline{z}\,\mathrm{d}z = \int_0^\pi (-2\cos(t)-2i\sin(t))(2\sin(t)+2i\cos(t))\,\mathrm{d}t = \int_0^\pi -4i(\sin^2(t)+\cos^2(t))\,\mathrm{d}t = \int_0^\pi -4i\,\mathrm{d}t = -4i\pi$$

Note that the function $\overline{z}$ is not analytic, a different contour with the same initial and terminal points could yield a different result.

### Fundamental theorem

Fundamental Theorem of Integration: If $f$ is **analytic** in a simply connected domain, $D$, and $z_0$ and $z_1$ are any two points joined by contour $C$ lying entirely in $D$, then,

$$\int_C f(z)\,\mathsf{d}z = F(z_1) - F(z_2)$$

In this situation, the function is independent of the path connecting the initial and terminal point.
Example: Compute $\int_C \cos(z)\,\mathsf{d}z$ where $C$ has the parameterization $z(t) = e^{i\pi t}, -\frac{1}{2} \leq t \leq \frac{1}{2}$.

$$\int_C \cos(z)\,\mathsf{d}z - \int_{-i}^{i} \cos(z)\,\mathsf{d}z = \sin(i) - \sin(-i) = \frac{-i}{2}(e^{-1} - e) + \frac{i}{2}(e - e^{-1}) = -ie^{-1} + ie$$

Simplified by recalling from Euler's Identity: $\sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ and $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

### ML-Inequality

If $f(z)$ is continuous on the contour $C$ then,

$$\left|\int_C f(z)\,\mathsf{d}z\right| \leq ML \qquad \text{where } |f(z)| \leq M \text{ on } C \text{ and } L \text{ is the length of } C.$$

Example:
If $f(z) = \frac{1}{z}$ and $C$ is the circle $|z| = r$ establish a bound for $\left|\int_C f(z)\,\mathsf{d}z\right|$.

$$|f(z)| \leq \frac{1}{r} \text{ and } L = 2\pi r$$

$$\left|\int_C f(z)\,\mathsf{d}z\right| \leq 2\pi$$

## Integrals over simple, closed contours

### Cauchy-Goursat theorem

If $f$ is analytic in a simply connected domain $D$ and $C$ is a simple, closed contour that lies in $D$, then,

$$\int_C f(z)\,\mathsf{d}z = 0$$

Example: $\int_C \frac{1}{z+1}\,\mathsf{d}z = 0$ for any contour that does not include $z = -1$ on its interior or on $C$.

How do we deal with more challenging integrals, where the antiderivative isn't known?

## Cauchy's integral formula

If $f$ is analytic in a simply connected domain $D$ and $C$ is a simple, closed contour that lies in $D$, and $z_0$ is any point inside $C$, then,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,\mathrm{d}z$$

And for derivatives,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z \qquad n = 1, 2, 3, \ldots$$

Example: Compute $\int_C \frac{e^{i\pi z}}{(4z-1)(z+2)}\,\mathrm{d}z$ over the positively oriented contour $C$: $|z| = 1$.

Notice that the only singularity $z = \frac{1}{4}$ is in the contour as $z = -2$ is outside the circle of radius 1. Identify the remainder of the integrand as $f(z)$ to apply Cauchy's Integral formula.

$$\int_C \frac{e^{2\pi iz}}{(4z-1)(z+2)}\,\mathrm{d}z = \int_C \frac{e^{2\pi iz}}{4(z-\frac{1}{4})(z-(-2))}\,\mathrm{d}z$$

$$= \int_C \frac{\frac{e^{2\pi iz}}{4(z+2)}}{z-\frac{1}{4}}\,\mathrm{d}z = \int_C \frac{f(z)}{z-\frac{1}{4}}\,\mathrm{d}z = 2\pi i f\left(\frac{1}{4}\right) = 2\pi\frac{e^{\frac{i\pi}{2}}}{-7} = \frac{2\pi\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)}{-7} = 2\pi i$$

Example: Compute $\int_C \frac{\cos(3z)}{z^3}\,\mathrm{d}z$ over the positively oriented ellipse $C : \frac{x^2}{9} + \frac{y^2}{4} = 0$.

Here $z_0 = 0$, a singularity that lies inside the contour, so using the theorem:

$$\int_C \frac{\cos(3z)}{z^3}\,\mathrm{d}z = \frac{2\pi i f^{(2)}(0)}{2!} = -9\pi i$$

Cauchy's Integral Formula provided us with a "trick" to integrate, allowing these examples to be completed without the need to parameterize the contour.

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