Fourier Series

This series is very important to many problems related to wave and heat analysis. It is used in solving ordinary and partial differential equations, often simplifying calculations.

Fourier Convergence Theorem: If $f(x)$ is piecewise smooth (*can have a finite number of discontinuities*) on [-L, L] then the Fourier series of f **converges** to $\frac{1}{2}(f(x+)+f(x-))$

For $f(x)$ on [-L, L], we want a_0, a_n, and b_n so that,

$$f(x)= a_0 + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

$n = 1, 2, \ldots$

In complex notation:

$$f(x)= \sum_{n=-\infty}^{\infty} d_n e^{inw_0 x} \qquad d_n = \frac{1}{L}\int_{-L}^{L} f(x) e^{inw_0 x} dx$$

Fourier Coefficients: You have done similar procedures before where you have represented functions as polynomials using Taylor Series approximations. Now, we want to represent $f(x)$ with an infinite sum of sines and cosines,

$$f(x) = \sum_{n=0}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})] \quad (1)$$

But how do we find the coefficients? To eventually get rid of the infinite sum, we will use the following orthogonality relations:

$$\int_{-L}^{L} \sin(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L})\, dx = 0$$

$$\int_{-L}^{L} \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L})\, dx = 0 \quad \text{if } m \neq n$$

$$\int_{-L}^{L} \sin(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L})\, dx = 0 \quad \text{if } m \neq n$$

Let's multiply (1) by $\cos(\frac{m\pi x}{L})$ and integrate the series term-by-term between –L and L,

$$\int_{-L}^{L} f(x) \cos(\frac{m\pi x}{L})\, dx = \sum_{n=0}^{\infty} \int_{-L}^{L} a_n \cos(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \cos(\frac{m\pi x}{L}) dx$$

$$\int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L})\, dx = \begin{cases} \int_{-L}^{L} a_0 \cos^2(0)\, dx = 2a_0 L & \text{if } n = m = 0 \\ \int_{-L}^{L} a_n \cos^2(\frac{n\pi x}{L})\, dx = a_n L & \text{if } n = m \neq 0 \end{cases}$$

Integrate using $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

Solving for the coefficients gives: $\mathbf{a_0} = \frac{1}{2L} \int_{-L}^{L} \boldsymbol{f(x)\, dx}$ and $\mathbf{a_n} = \frac{1}{L} \int_{-L}^{L} \boldsymbol{f(x) \cos(\frac{n\pi x}{L})\, dx}$

Similarly, we can multiply both sides of (1) by $\sin(\frac{m\pi x}{L})$ and integrate on $-L \leq x \leq L$ to obtain,

$$\boldsymbol{b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L})\, dx}$$

Now we have the ability to find the Fourier series representation of a function.

Example: Find the Fourier series for the given function. $f(x) = \begin{cases} 1 & -1 \le x < 0 \\ x & 0 \le x < 1 \end{cases}$

Compute the coefficients:

$$a_n = \frac{1}{L}\int_{-1}^{1} f(x)\cos(m\pi x)\,dx$$

$$= \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1} x\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} + \frac{x}{n\pi}\sin(n\pi x)\Big|_{0}^{1} - \int_{0}^{1}\frac{1}{n\pi}\sin(n\pi x)\,dx$$

$$= \frac{1}{n^2\pi^2}(\cos(n\pi) - 1)$$

$$= \frac{1}{n^2\pi^2}((-1)^n - 1)$$

Integration by parts

Recall: $\sin(n\pi) = 0$
$\cos(n\pi) = (-1)^n$

$$a_0 = \frac{1}{2L}\int_{-1}^{1} f(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{0} dx + \frac{1}{2}\int_{0}^{1} x\,dx$$

$$= \frac{3}{4}$$

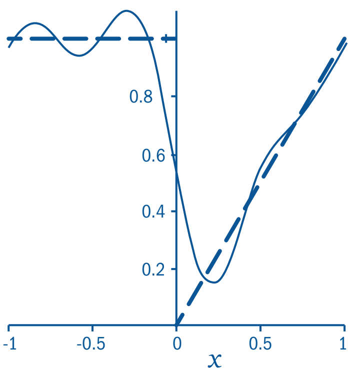
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_{-1}^{0}\sin(n\pi x)\,dx + \int_{0}^{1} x\sin(n\pi x)\,dx$$

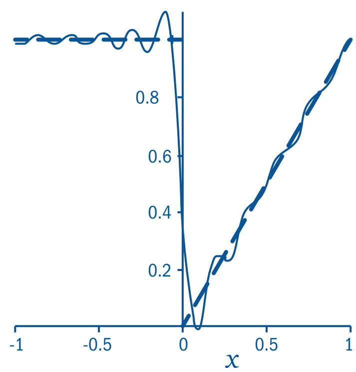
$$= \frac{-1 + \cos(n\pi)}{n\pi} + \frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2\pi^2}$$

$$= \frac{-1}{n\pi}$$

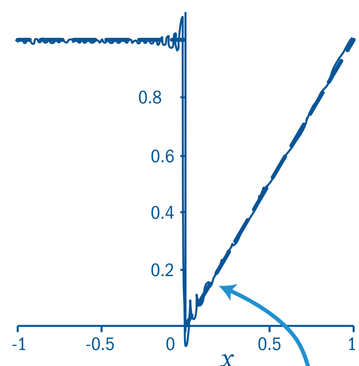
Therefore the Fourier series is $f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{((-1)^n - 1)}{n^2\pi^2}\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)\right]$



$n = 3$



$n = 10$



$n = 50$

The "ringing" at the jump discontinuities is called Gibbs phenomena

If the function has symmetry, some of the computations become easier

Recall from calculus:

If $f(x)$ is even, $\int_{-L}^{L} f(x) = 2\int_{0}^{L} f(x)\,dx$

If $f(x)$ is odd, $\int_{-L}^{L} f(x) = 0$

The product of two odd functions is even.
The product of two even function is even.
The product of an odd and an even function is odd.

If a function is odd, its Fourier series will be of the form $\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$ (Since $a_0 = a_n = 0$)

If a function is even, its Fourier series will be of the form $a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$ (Since $b_n = 0$)

Exercise: Find the Fourier series representation of $f(x) = x^2$ on $-2 \le x \le 2$.

Solution: Here, $L = 2$ so $f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\frac{16(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)$

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