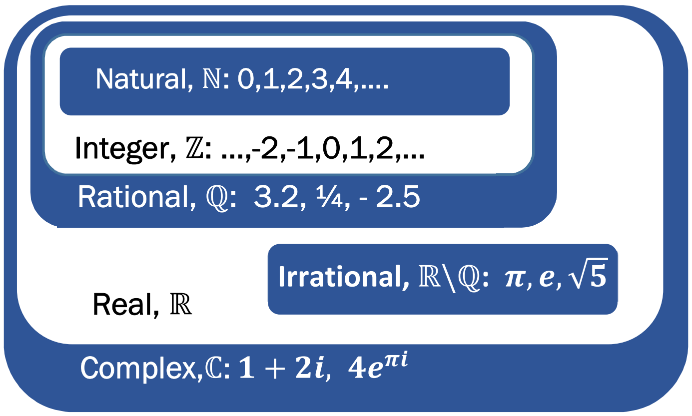
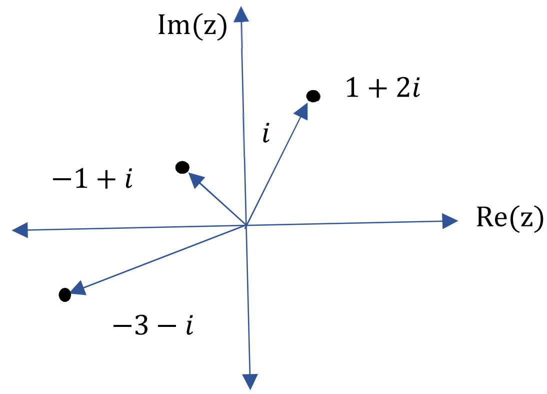
Complex numbers

The complex number system is applied to solve and simplify many math problems ranging from trigonometry and geometry to algebra and calculus.





Cartesian form

$z = x + iy = Re(z) + iIm(z)$

Arithmetic	Useful results
Addition/Subtraction: $z_1 \pm z_2 = Re(z_1) \pm Re(z_2) + i(\text{Im}(z_1) \pm Im(z_2))$ Example: $(-1 + i) + (1 + 2i) = (-1 + 1) + (1 + 2)i = 3i$	$\frac{1}{z} = \frac{\bar{z}}{\lvert z\rvert^2}$ $z\bar{z} = \lvert z\rvert^2$ $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ $\frac{1}{i} = -i$
Multiplication: use the distributive property and $i^2 = -1$ Example: $(-1 + i)(1 + 2i) = -1 - 2i + i + 2i^2 = -1 - i - 2 = -3 - i$	
Division: rationalize (multiply by the conjugate of the denominator) Example: $\frac{1+i}{1+2i} = \frac{1+i}{1+2i}\left(\frac{1-2i}{1-2i}\right) = \frac{1-i-2i^2}{1+4} = \frac{3}{5} - \frac{1}{5}i$	

Polar form

$z = re^{i\theta} = |z|e^{iArg(z)}$

Modulus (notion of length, or distance from zero): $|z| = \sqrt{x^2 + y^2}$
Principle Argument: $Arg\ z = \theta$ where $\theta \in (-\pi, \pi]$
This can be solved for using $\tan(\theta) = \frac{y}{x}$. Remember to take into consideration the signs of both x and y to determine the correct quadrant for θ.
Argument: $\arg z = \{\theta + 2\pi k : k = 0, \pm1, \pm2, \dots\}$

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In practice, use whichever complex form is most convenient. Addition and subtraction are easier in Cartesian form whereas multiplication and division are easier in polar form.

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Example: Convert $-1+i$ to polar form.

$$|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$

$$Arg(z) = \frac{3\pi}{4}$$

$$-1+i = \sqrt{2}e^{i\frac{3\pi}{4}}$$

Note: $\arctan(-1) = -\frac{\pi}{4}$ but the signs of x and y indicate the angle is in Quadrant II so the corresponding angle is $\frac{3\pi}{4}$.

Euler's identity

$$e^{ir\theta} = r(\cos(\theta) + isin(\theta))$$

Example: Convert $2e^{i\frac{5\pi}{6}}$ into Cartesian Form.

Using Euler's identity,

$$2e^{i\frac{5\pi}{6}} = 2\left(\cos\left(\frac{5\pi}{6}\right) + isin\left(\frac{5\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -\sqrt{3} + i$$

Powers and roots

De Moivre's Formula: $z^n = r^n e^{n\theta i} = r^n(\cos(n\theta) + isin(n\theta))$

Roots are bit more complicated since there are multiple solutions that need to be obtained:

$$z^{1/n} = |z|^{1/n} e^{\frac{i(\theta+2k\pi)}{n}}, \text{ for } k = 0, 1, 2, \ldots n-1$$

Example: Find the cube roots of $1 + i\sqrt{3}$.

First convert to polar form: $1 + \sqrt{3} = 2e^{i\frac{\pi}{3}}$

$$\left(1+\sqrt{3}\right)^{1/3} = 2^{1/3} e^{i\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)} \text{ for } k = 0, 1, 2$$

So the three cube roots of $1 + i\sqrt{3}$ are $2^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{9}\right) + isin\left(\frac{\pi}{9}\right)\right)$, $2^{\frac{1}{3}}\left(\cos\left(\frac{7\pi}{9}\right) + isin\left(\frac{7\pi}{9}\right)\right)$, and $2^{\frac{1}{3}}\left(\cos\left(\frac{13\pi}{9}\right) + isin\left(\frac{13\pi}{9}\right)\right)$