Complex integration

If $f(t) = u(t) + iv(t)$, where t is a real variable, we can extend our idea of integration easily as follows:

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)\,dt$$

Example: $\int_0^1 (2t+i)^3\,dt = \int_i^{2+i} \frac{1}{2}\,u^3\,du = \frac{(2+i)^4}{8} - \frac{i^4}{8} = \frac{(2+i)^4+1}{8}$

Contour integrals

In order to integrate a complex mapping $f(z)$ from a to b we need to a have a specified curve to follow.

Rules for integration

$$\int_C af(z)dz = a\int_C f(z)z$$

$$\int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

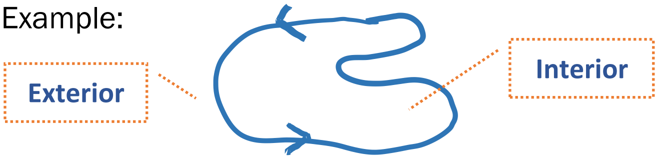
$$\int_{C+K} f(z)dz = \int_C f(z)dz + \int_K f(z)dz$$

$$\int_{-C} f(z)dz = -\int_C f(z)z$$

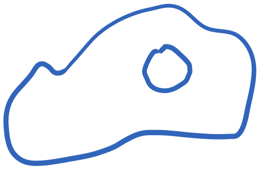
Important terms to know

Simply connected: any simple (does not cross itself) closed contour without holes.

Example:



Non-Example:



If the contour, C, is positively orientated that means the interior is on the left and exterior on the right.

Important contours to know

A **line segment** from $z = z_0 \; to \; z = z_1$:

$$z(t) = z_0 + t(z_1 - z_0), 0 \le t \le 1$$

A **circle** of radius $R > 0$, centre at z_0:

$$z(t) = z_0 + Re^{it}, \theta_0 \le t \le \theta_0 + 2\pi \quad \text{(counter-clockwise)}$$

$$z(t) = z_0 + Re^{i(2\pi+\theta_0-t)}, \theta_0 \le t \le \theta_0 + 2\pi \quad \text{(clockwise)}$$

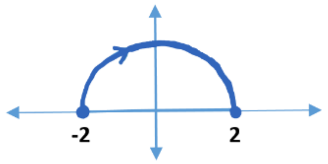
Examples (other parametrizations of the same curve are possible):

$z(t) = i + 2e^{i\pi t}, 0 \le t \le 1$	a semi-circle, radius 2, centre i, counterclockwise
$z(t) = t + i, -1 \le t \le 1$	a line segment from -1 $+ i$ to $1 + i$
$z(t) = 4 + e^{i(2\pi-t)}, 0 \le t \le 2\pi$	a circle, radius 1, centre 4, clockwise

If $f(z)$ is defined on a set including the contour, C, integrating along the contour can be done as follows:

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\, z'(t)dt, \;\; a \le t \le b$$

Example: Compute $\int_C \bar{z}\; dz$ on the contour shown.



First, parameterize the contour. $C: z(t) = 2e^{i(\pi-t)}, 0 \le t \le \pi$

Using Euler's Identity this can be rewritten as:

$$2e^{i(\pi-t)} = 2\cos(\pi - t) + 2i\sin(\pi - t) = -2\cos(t) + 2i\sin(t)$$

$$f\big(z(t)\big) = \overline{-2\cos(t) + 2i\sin(t)} = 2\cos(\pi - t) - 2i\sin(\pi - t) = -2cost(t) - 2i\; sin(t)$$

$$z'(t) = 2\sin(t) + 2i\; cos(t)$$

$$\int_C \bar{z}\; dz = \int_0^\pi (-2\cos(t) - 2i\sin(t))\big(2\sin(t) + 2i\; cos(t)\big)dt = \int_0^\pi -4i(\sin^2 t + \cos^2 t)\; dt$$
$$= \int_0^\pi -4i\; dt = -4i\pi$$

Note that the function $\bar{z}$ is not analytic, a different contour with the same initial and terminal points could yield a different result.

Fundamental theorem

Fundamental Theorem of Integration: If f is **analytic** in a simply connected domain, D, and z_0 and z_1 are any two points joined by contour C lying entirely in D, then

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

In this situation, the function is independent of the path connecting the initial and terminal point.

Example: Compute $\int_C \cos(z)\, dz$ where C has the parametrization $z(t) = e^{i\pi t}, -\frac{1}{2} \leq t \leq \frac{1}{2}$.

$$\int_C \cos(z)\, dz = \int_{-i}^{i} \cos(z)\, dz = \sin(i) - \sin(-i) = \frac{-i}{2}(e^{-1} - e) + \frac{i}{2}(e - e^{-1}) = -ie^{-1} + ie$$

Simplified by recalling from Euler's Identity: $\sin(\theta) = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$ and $\cos(\theta) = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

ML- Inequality

If f(z) is continuous on the contour C then,

$\left|\int_C f(z)dz\right| \leq ML$ where $|f(z)| \leq M$ on C and L is the length of C

Example:

If $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ establish a bound for $\left|\int_C f(z)dz\right|$.

$|f(z)| \leq \frac{1}{r}$ and $L = 2\pi r$

$\left|\int_C f(z)dz\right| \leq 2\pi$

Integrals over simple, closed contours

Cauchy-Goursat theorem

If f is analytic in a simply connected domain, D, and C is a simple, closed contour that lies in D, then,

$$\int_C f(z)dz = 0$$

Example: $\int_C \frac{1}{z+1}\, dz = 0$ for any contour that does not include $z = -1$ on its interior or on C.

How do we deal with more challenging integrals, where the antiderivative isn't known?

Cauchy's integral formula

If f is analytic in a simply connected domain, D, and C is a simple, closed contour that lies in D, and $z_0$is any point inside C, then,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz$$

And for derivatives,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \qquad n = 1, 2, 3, \ldots$$

Example: Compute $\int_C \frac{e^{i\pi z}}{(4z-1)(z+2)}\, dz$ over the positively oriented contour $C\colon\ |z| = 1$

Notice that only the singularity $z = \frac{1}{4}$ is in the contour as $z = -2$ is outside the circle of radius 1. Identify the remainder of the integrand as $f(z)$ to apply Cauchy's Integral formula.

$$\int_C \frac{e^{i2\pi z}}{(4z-1)(z+2)}\, dz = \int_C \frac{e^{i2\pi z}}{4\left(z-\frac{1}{4}\right)\left(z-(-2)\right)}\, dz$$

$$= \int_C \frac{\frac{e^{i2\pi z}}{4(z+2)}}{\left(z-\frac{1}{4}\right)}\, dz = \int_C \frac{f(z)}{\left(z-\frac{1}{4}\right)}\, dz = 2\pi i f\left(\frac{1}{4}\right) = 2\pi\frac{e^{\frac{i\pi}{2}}}{-7} = \frac{2\pi\left(\cos\left(\frac{\pi}{2}\right)+i\, sin\left(\frac{\pi}{2}\right)\right)}{-7} = 2\pi i$$

Example: Compute $\int_C \frac{\cos(3z)}{z^3}\, dz$ over the positively oriented ellipse $C\colon \frac{x^2}{9} + \frac{y^2}{4} = 0$

Here $z_0 = 0$, a singularity that lies inside the contour, so using the theorem:

$$\int_C \frac{\cos(3z)}{z^3}\, dz = \frac{2\pi i\, f^{(2)}(0)}{2!} = -9\pi i$$

Cauchy's Integral Formula provided us with a "trick" to integrate, allowing these examples to be completed without the need to parameterize the contour.