



Complex functions

A complex function can be written in terms of its real and imaginary parts:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y) \quad \text{or} \quad w = f(z) = f(re^{i\theta}) = u(r, \theta) + i v(r, \theta)$$

Exponential function

Power series: $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$

Some similarities with its real counterpart include: $e^0 = 1$, $e^z \neq 0$, $\frac{d}{dz} e^z = e^z$, $e^{z_1} e^{z_2} = e^{z_1+z_2}$

Some notable differences: e^z is not one-to-one, e^z can attain negative values

A key property used for computation is $e^z = e^x(\cos y + i \sin y)$, hence e^z is periodic:

$$e^z \equiv 1 \quad \text{if } z = 2k\pi i, k \in \mathbb{Z}$$
$$e^{z_1} \equiv e^{z_2} \quad \text{if } z_1 = z_2 + 2k\pi i, k \in \mathbb{Z}$$

Example: $e^{2+i\frac{\pi}{4}} = e^2(\cos(\frac{\pi}{4}) + i \sin(\frac{\pi}{4})) = e^2(\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2})$

Example: Find all $z \in \mathbb{C}$ such that $e^z = \sqrt{3} - i$

First, write in polar form: $|\sqrt{3} - i| = \sqrt{3+1} = 2$ and $Arg(\sqrt{3} - i) = -\frac{\pi}{6}$

Then, $e^z = e^{x+iy} = e^x e^{iy} = 2e^{-i\frac{\pi}{6}}$ only when:

$$e^x = 2 \quad \text{and} \quad e^{iy} = e^{-\frac{i\pi}{6}} \quad \text{or} \quad e^{iy - (-\frac{\pi}{6})} = 1$$

Which is true when $x = \ln(2)$ and $y + \frac{\pi}{6} = 2\pi k$ or $y = \frac{\pi}{6}(12k - 1), k \in \mathbb{Z}$

Notation: $f(z) = e^{\frac{1}{3}}$ may be interpreted as any of the complex third roots so sometimes $\exp(z)$ is used to denote that a single value is chosen, i.e. $\exp(\frac{1}{3}) \approx 1.3956$ whereas $e^{1/3} \approx 1.3956(\cos(\frac{2\pi k}{3}) + i \sin(\frac{2\pi k}{3}))$ for $k = 0, 1, 2$.

Logarithmic function

Multivalued logarithm:

$$\log(z) = \{\ln|z| + i(Arg(z) + 2n\pi)\} = \{\ln(z) + iarg(z)\}, \text{ where } n \text{ is an integer and for } z \neq 0$$

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Single-valued logarithm (principal value):

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z), \text{ for } z \neq 0,$$

Example: $\text{Log}(\sqrt{3} - i) = \ln(2) + i\left(-\frac{\pi}{6}\right)$

Properties:

$$\exp(\text{Log}(z)) = z \text{ for } z \neq 0 \quad \& \quad \text{Log}(\exp(z)) = z \text{ for } -\pi < \text{Im}(z) \leq \pi \quad (\text{Recall: } \text{Arg}(z) \in (-\pi, \pi])$$

It also gives a way to define complex powers, for $\alpha \in \mathbb{C}$:

$$z^\alpha = e^{\alpha \log(z)} = e^{\alpha(\ln|z| + i \arg(z))}$$

Note: if α is rational, only a finite number of z^α result

Example: Determine all possible values of $(i)^{\sqrt{3}/4}$.

First, if $z = i$, then $|z| = \sqrt{0^2 + 1^2} = 1$ and $\text{Arg}(z) = \frac{\pi}{2}$

$$\begin{aligned} z^\alpha &= \left\{ e^{\frac{\sqrt{3}}{4}(\ln(1) + i(\frac{\pi}{2} + 2\pi n))} \mid n \in \mathbb{Z} \right\} \\ &= \left\{ e^{\frac{\sqrt{3}}{4}i(\frac{\pi}{2} + 2\pi n)} \mid n \in \mathbb{Z} \right\} = \left\{ e^{i\frac{3\pi}{8}(1+4n)} \mid n \in \mathbb{Z} \right\} \\ &= \left\{ \cos\left(\frac{\sqrt{3}\pi}{8}(1+4n)\right) + i \sin\left(\frac{\sqrt{3}\pi}{8}(1+4n)\right) \mid n \in \mathbb{Z} \right\} \end{aligned}$$

Example: Determine all possible values of $(i + \sqrt{3})^i$.

Ans: $\left\{ e^{-\frac{\pi}{6} + 2\pi n} [\cos(\ln 2) + i \sin(\ln 2)] \mid n \in \mathbb{Z} \right\}$

Note: The Maple command `evalc` yields the principal value.

Trigonometric functions

The familiar identities and derivatives for trigonometric functions still hold in \mathbb{C} .

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos z = \frac{1}{2i}(e^{iz} + e^{-iz})$$

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y, \quad \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \quad \cosh z = \frac{1}{2}(e^z + e^{-z})$$

A notable difference: $\sin z$ and $\cos z$ are unbounded, unlike $|\sin x| \leq 1, x \in \mathbb{R}$.

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \quad |\cos z|^2 = \cos^2 x + \sinh^2 y$$

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