Residues and residue theorem

Residues definition

Let f have a **nonremovable isolated singularity** at z_0 and the Laurent series, $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$

The residue is:

$$Res[f, z_0] = a_{-1} \text{ where } a_{-1} \text{ is the coefficient of the } \frac{1}{z-z_0} \text{ term of the Laurent series}$$

Example: Find the $Res[f, z_0]$ if $f(z) = e^{\frac{3}{z-1}}$.

The Laurent series is: $e^{\frac{1}{z-1}} = 1 + 3\left(\frac{1}{z-1} + \frac{3^2}{2!(z-1)^2} + \frac{3^3}{3!(z-1)^3} + \cdots\right.$

So, $Res[f, 1] = 3$

Calculating residues at poles

Expanding the Laurent series to identify the coefficient is a general method that will work to find the residues of essential singularities and poles, but if z_0 is a pole of degree k of $f(z)$ it can be easier to use the following equations find the residue:

If z_0 is a simple pole, then $Res[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z)$

If z_0 is a pole of order k, then $Res[f, z_0] = \lim_{z \to z_0} \left[\frac{d^{k-1}}{dz^{k-1}}\left(\frac{(z-z_0)^k f(z)}{(k-1)!}\right)\right]$

Try this method to calculate the residue in the previous example.

Cauchy's residue theorem

This simplifies the calculation of contour integrals (over simple closed contours) to just computing residues. It is an extension of the Cauchy integral formula, which helped us evaluate integrands of the form $\frac{f(z)}{(z-z_0)^k}$. It is used to evaluate problems with a finite number of isolated singularities.

If C is a simple, closed, positively oriented contour and f is analytic inside and on C except at points z_j then,

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{n} Res(z_j)$$

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Example: Compute $\int_C f(z)\ dz = \int_C \frac{1-z}{(z-4)(z+2)^3}\ dz$ over the positively oriented contour $C\colon\ |z| = 5$

The integrand has a simple pole at $z_0 = 4$ and a pole of order 3 at $z_0 = -2$. Both of these points lie within the contour so by Residue Theorem:

$$Res[f,4] = \lim_{z\to 4}(z-4)\frac{1-z}{(z-4)(z+2)^3} = \lim_{z\to 4}\frac{1-z}{(z+2)^3} = \frac{-3}{6^3} = \frac{-1}{72}$$

$$Res[f,-2] = \lim_{z\to -2}\frac{d^2}{dz^2}\left(\frac{(z+2)^3}{2!}\right)\left(\frac{1-z}{(z-4)(z+2)^3}\right) = \lim_{z\to -2}\frac{1}{(z-4)^2} + \frac{1-z}{(z-4)^3} = \frac{1}{72}$$

$$\int_C \frac{1-z}{(z-4)(z+2)^3}\ dz = 2\pi i\left(-\frac{1}{72} + \frac{1}{72}\right) = 0$$

Example: Compute $\int_C \frac{\sin(3z)}{z^2(z-1)}\ dz$ over the positively oriented ellipse $C\colon \frac{x^2}{9} + \frac{y^2}{4} = 0$

Here $z_0 = 0$ is a pole of order 2 that lies within the contour and a simple pole ($z_0 = 1$) that lies outside the contour. So by Residue Theorem,

$$Res[f,0] = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^2}{1!}\right)\left(\frac{\sin(3z)}{z^2(z-1)}\right) = \lim_{z\to 0}\left(\frac{3\cos(3z)}{\mathrm{z}-1}\right) = -3$$

$$Res[f,1] = \lim_{z\to 1}(z-1)\frac{\sin(3z)}{z^2(z-1)} = \sin(3)$$

$$\int_C \frac{\sin(3z)}{z^2(z-1)}\ dz = 2\pi i(-3 + \sin(3))$$